

Equazioni fotonica

$$\left\{ \begin{array}{l} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \frac{\omega^2}{c^2} \epsilon(r) \vec{E} \\ \vec{H}(\vec{r}) = \frac{-i}{\omega \mu} \vec{\nabla} \times \vec{E}(\vec{r}) \end{array} \right. \quad \begin{array}{l} \vec{\nabla} \cdot \epsilon(\vec{r}) \vec{E} = 0 \\ (\vec{E}_i(\vec{r}), \epsilon(\vec{r}) \vec{E}_j(\vec{r})) = A_i \delta_{i,j} \end{array}$$

$$\left\{ \begin{array}{l} \vec{\nabla} \times \left(\frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times \vec{H} \right) = \frac{\omega^2}{c^2} \vec{H} \\ \vec{E}(\vec{r}) = \frac{i}{\omega \epsilon_o \epsilon(\vec{r})} \vec{\nabla} \times \vec{H}(\vec{r}) \end{array} \right. \quad \begin{array}{l} \vec{\nabla} \cdot \vec{H} = 0 \\ (\vec{H}_i(\vec{r}), \vec{H}_j(\vec{r})) = (\mu_0 c)^2 A_j \delta_{i,j} \end{array}$$

Autostati e autovalori del problema iniziale

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}_i(\vec{r})) = \frac{\omega_i^2}{c^2} \epsilon(\vec{r}) \vec{E}_i(\vec{r})$$

Perturbazione

$$\tilde{\epsilon}(\vec{r}) = \epsilon(\vec{r}) + \delta\epsilon(\vec{r})$$

Soluzione generale

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}(\vec{r})) = \frac{\omega^2}{c^2} \tilde{\epsilon}(\vec{r}) \vec{E}(\vec{r})$$

Stima variazione n-esimo autostato (hp piccola perturbazione)

$$\vec{E}(\vec{r}) = c_n \vec{E}_n(\vec{r}) + \sum_{i \neq n} \delta c_i \vec{E}_i(\vec{r}) \quad \delta c_i \ll c_n$$

Stima variazione n-esimo autovalore

$$\frac{\omega_n^2}{c^2} \epsilon(\vec{r}) c_n \vec{E}_n(\vec{r}) + \frac{\omega_i^2}{c^2} \epsilon(\vec{r}) \sum_{i \neq n} \delta c_i \vec{E}_i(\vec{r}) =$$

$$= \frac{\omega^2}{c^2} (\epsilon(\vec{r}) + \delta\epsilon(\vec{r})) \left[c_n \vec{E}_n(\vec{r}) + \sum_{i \neq n} \delta c_i \vec{E}_i(\vec{r}) \right]$$

Metodo perturbativo

Proiezione sullo stato n-esimo

Metodo perturbativo

$$\begin{aligned} \frac{\omega_n^2}{c^2} c_n (\vec{E}_n(\vec{r}), \mathcal{E}(\vec{r}) \vec{E}_n(\vec{r})) + \frac{\omega_i^2}{c^2} \sum_{i \neq n} \delta c_i (\vec{E}_n(\vec{r}), \mathcal{E}(\vec{r}) \vec{E}_i(\vec{r})) = \\ = \frac{\omega^2}{c^2} \left[c_n (\vec{E}_n(\vec{r}), (\mathcal{E}(r) + \delta \mathcal{E}(\vec{r})) \vec{E}_n(\vec{r})) + \sum_{i \neq n} \delta c_i (\vec{E}_n(\vec{r}), \mathcal{E}(r) \vec{E}_i(\vec{r})) \right] \end{aligned}$$

Ricordando che per autostati vale:

$$(\vec{E}_n(\vec{r}), \mathcal{E}(\vec{r}) \vec{E}_i(\vec{r})) = (\vec{E}_n(\vec{r}), \mathcal{E}(\vec{r}) \vec{E}_n(\vec{r})) \delta_{n,i} \equiv A_n \delta_{n,i}$$

Si ha:

$$\frac{\omega_n^2}{c^2} c_n A_n = \frac{\omega^2}{c^2} c_n (A_n + (\vec{E}_n(\vec{r}), \delta \mathcal{E}(\vec{r}) \vec{E}_n(\vec{r})))$$

Quindi

Metodo perturbativo

$$\frac{(\omega_n^2 - \omega^2)}{\omega^2} = \frac{(\vec{E}_n(\vec{r}), \delta\epsilon(\vec{r})\vec{E}_n(\vec{r}))}{(\vec{E}_n(\vec{r}), \epsilon(\vec{r})\vec{E}_n(\vec{r}))}$$

Ponendo: $\Delta\omega_n = \omega - \omega_n$ e se $|\Delta\omega_n| \ll \omega_n$

Si ottiene:

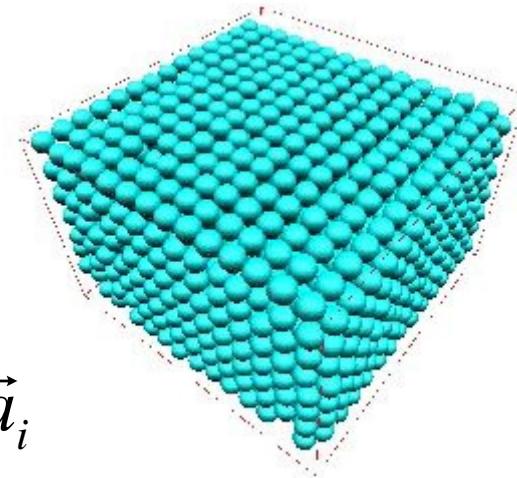
$$\Delta\omega_n = -\frac{\omega_n}{2} \frac{\int d^3r \vec{E}_n^*(\vec{r}) \delta\epsilon(\vec{r}) \vec{E}_n(\vec{r})}{\int d^3r \vec{E}_n^*(\vec{r}) \epsilon(\vec{r}) \vec{E}_n(\vec{r})}$$

Un aumento di dielettrico produce un red shift

Una diminuzione di dielettrico produce un blue shift

Teorema di Bloch

$$\hat{T}_{\vec{R}} \mathcal{E}(\vec{r}) = \mathcal{E}(\vec{r} + \vec{R}) = \mathcal{E}(\vec{r}) \quad \vec{R} = \sum \ell_i \vec{a}_i$$



$$\hat{\Theta} \vec{H}(\vec{r}) = \frac{\omega^2}{c^2} \vec{H}(\vec{r}) \quad \hat{T}_{\vec{R}} \vec{H}(\vec{r}) = e^{i\varphi(\vec{R})} \vec{H}(\vec{r})$$

$$[\hat{T}_{\vec{R}}, \hat{\Theta}] \vec{H}(\vec{r}) = \hat{T}_{\vec{R}} \hat{\Theta} \vec{H}(\vec{r}) - \hat{\Theta} \hat{T}_{\vec{R}} \vec{H}(\vec{r}) = 0$$

$$\vec{H}(\vec{r}) \equiv \vec{H}_{n,\vec{k}}(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} \vec{u}_{n,\vec{k}}(\vec{r}) \quad \vec{u}_{n,\vec{k}}(\vec{r} + \vec{R}) = \vec{u}_{n,\vec{k}}(\vec{r})$$

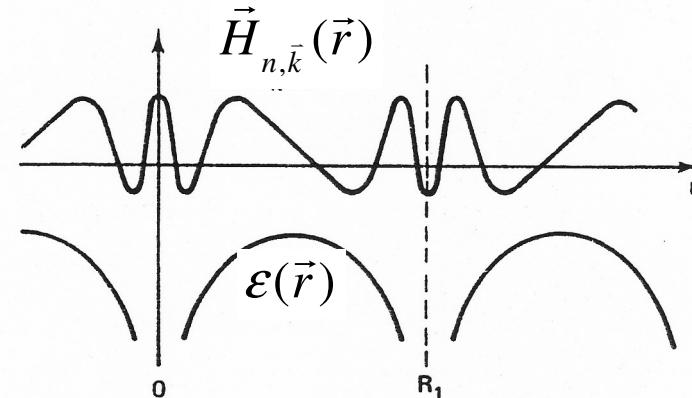
n=indice di banda (numera i modi in ordine di ω crescente)

k=vettore onda modo fotonico

Prima zona di Brillouin (FBZ)

Th. Bloch

$$\vec{H}_{n,\vec{k}}(\vec{r} + \vec{R}) = \vec{H}_{n,\vec{k}}(\vec{r}) e^{i\vec{k} \cdot \vec{R}}$$



Periodicità $\vec{K} \cdot \vec{R} = 2n\pi$

$$\vec{H}_{n,\vec{k}+\vec{K}}(\vec{r} + \vec{R}) = e^{i(\vec{k}+\vec{K}) \cdot \vec{r}} e^{i\vec{k} \cdot \vec{R}} \vec{u}_{n,\vec{k}+\vec{K}}(\vec{r}) =$$

$$= \vec{H}_{n,\vec{k}+\vec{K}}(\vec{r}) e^{i\vec{k} \cdot \vec{R}} = \vec{H}_{n,\vec{k}}(\vec{r} + \vec{R})$$

$$\boxed{\vec{k} + \vec{K} \equiv \vec{k}}$$

First Brillouin Zone

- Il vettore d'onda definisce univocamente un modo solo nella FBZ
- Tutti i modi sono rappresentati nella FBZ

Time-reversal

$$\hat{\Theta} \vec{H}_{n,\vec{k}}(\vec{r}) = \frac{\omega_n^2(\vec{k})}{c^2} \vec{H}_{n,\vec{k}}(\vec{r})$$

$$[\hat{\Theta} \vec{H}_{n,\vec{k}}(\vec{r})]^* = \hat{\Theta} \vec{H}_{n,\vec{k}}^*(\vec{r}) = \frac{\omega_n^2(\vec{k})}{c^2} \vec{H}_{n,\vec{k}}^*(\vec{r})$$

$$\vec{H}_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \vec{u}_{n,\vec{k}}(\vec{r})$$

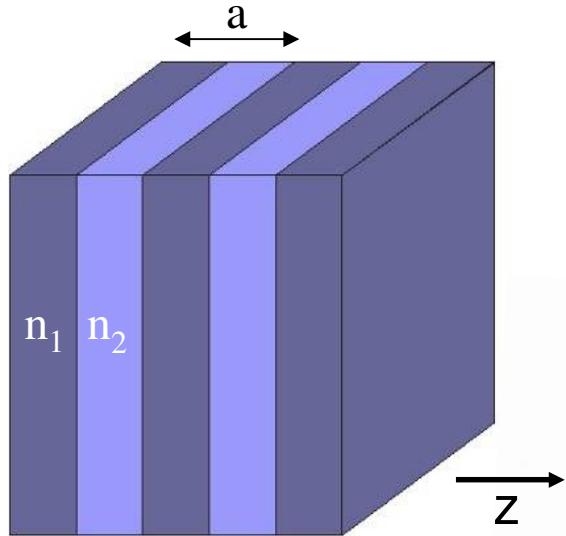
$$\vec{H}_{n,\vec{k}}^*(\vec{r}) = e^{-i\vec{k}\cdot\vec{r}} \vec{u}_{n,\vec{k}}^*(\vec{r}) \equiv \vec{H}_{n,-\vec{k}}(\vec{r})$$

$$\omega_n(\vec{k}) = \omega_n(-\vec{k})$$

Le bande sono simmetriche in k

PhC in 1D

1D

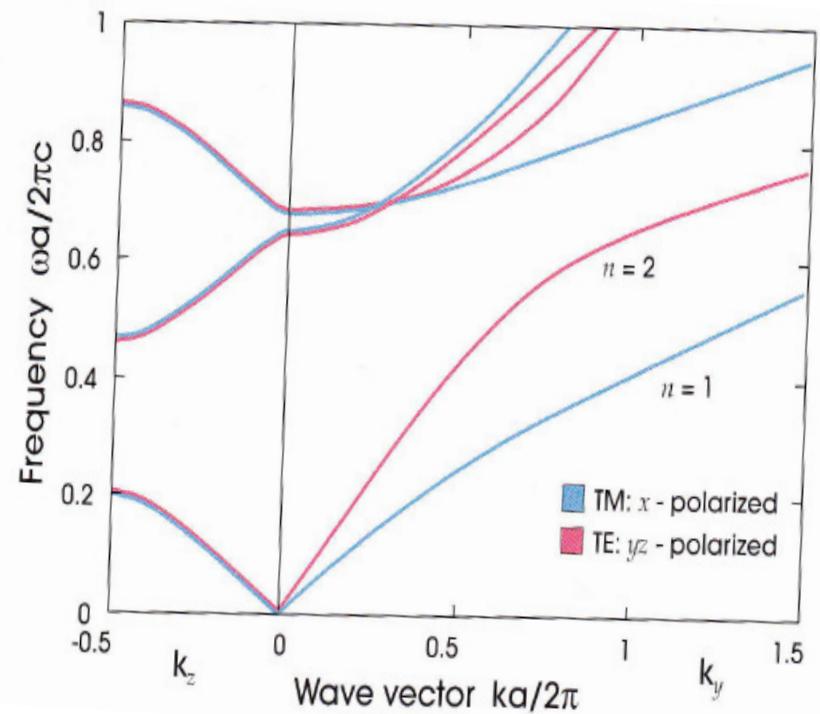
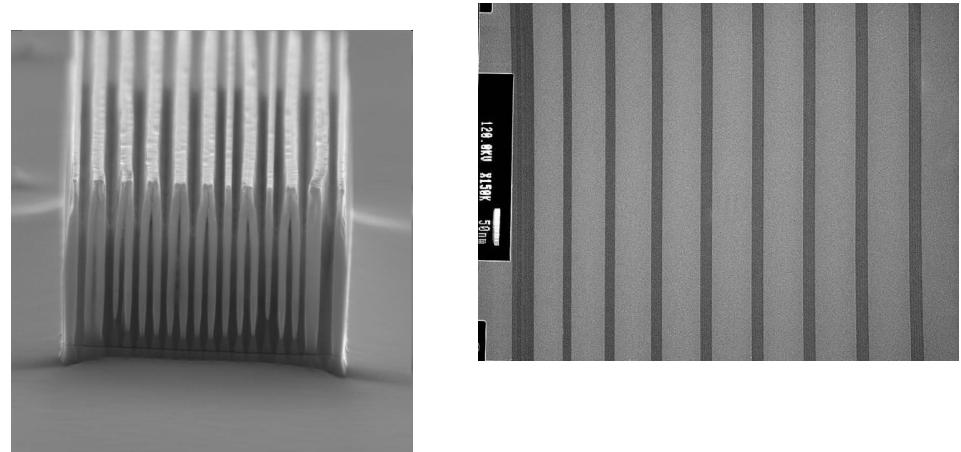


$$\vec{H}_{n,\vec{k}}(\vec{r}) = e^{ik_z z} e^{i\vec{k}_{||} \cdot \vec{\rho}} \vec{u}_{n,k_z}(z)$$

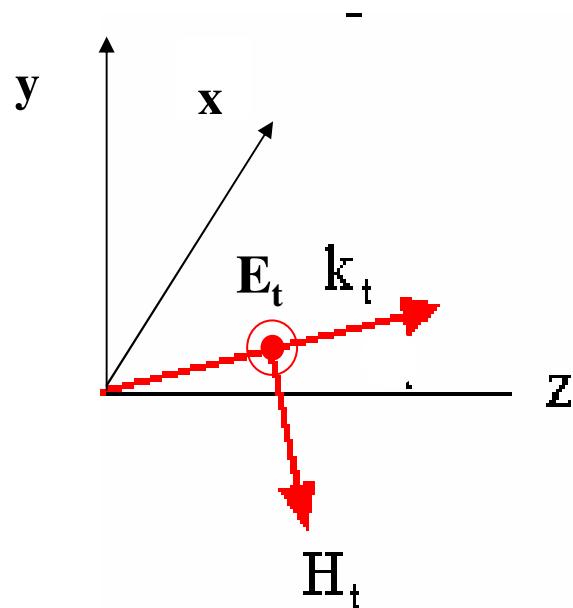
$$\vec{k} = \vec{k}_{||} + \vec{k}_z$$

$$\vec{\rho} = x\hat{x} + y\hat{y}$$

$$FBZ \quad -\frac{\pi}{a} \leq |\vec{k}_z| \leq \frac{\pi}{a}$$



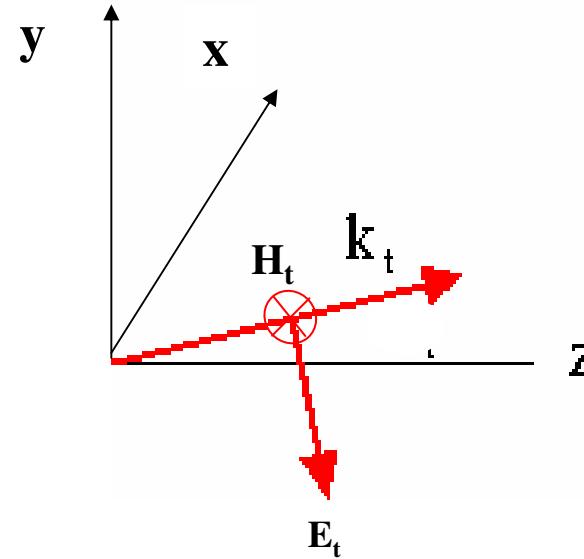
Nota su onde TE e TM (attenzione!)



Ottica classica

Onda s (senkrecht)

Polarizzazione TE



Onda s (parallel)

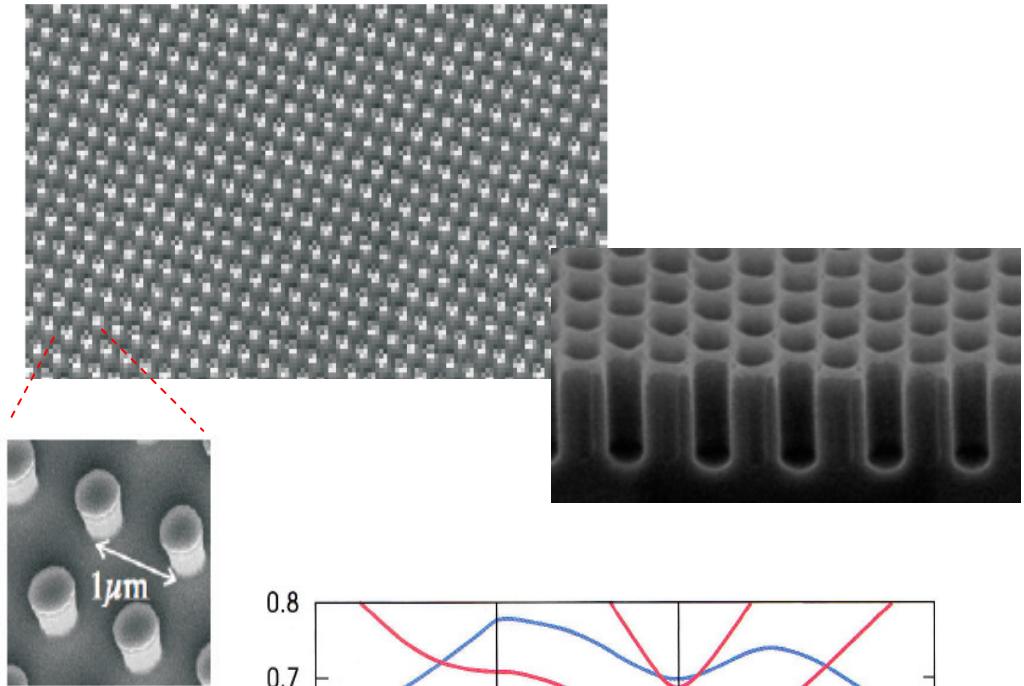
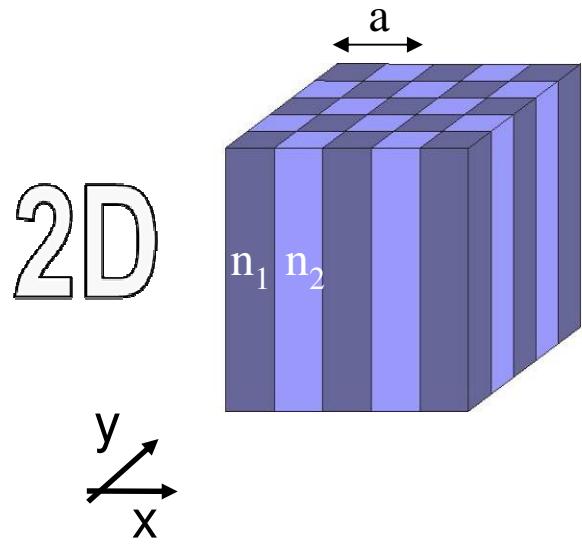
Polarizzazione TM

Fotonica (Joannopoulos)

Polarizzazione TM

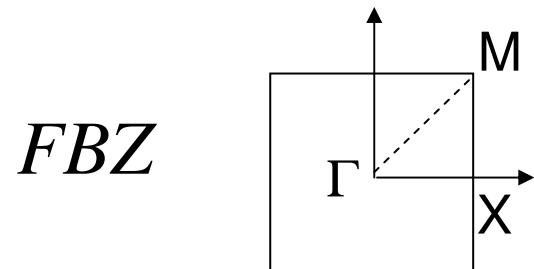
Polarizzazione TE

PhC in 2D

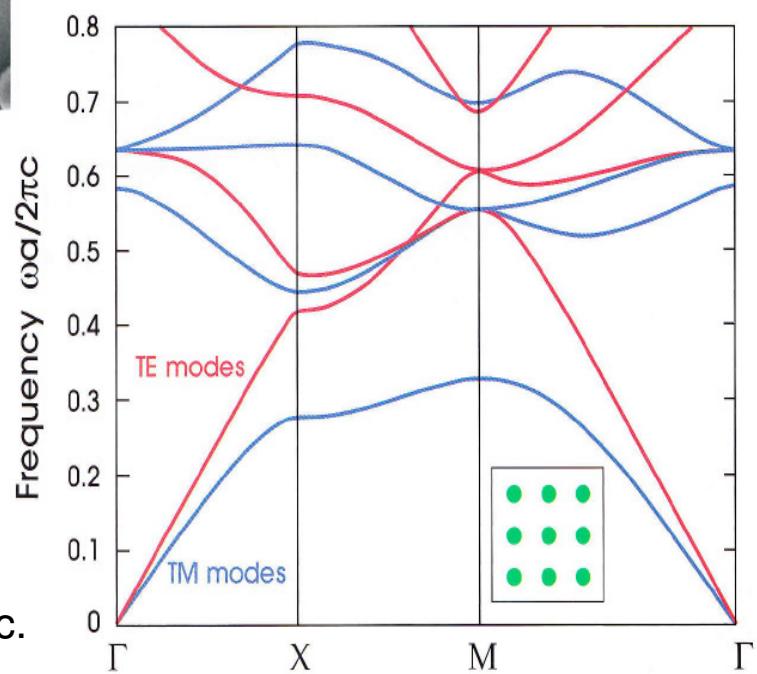


$$\vec{H}_{n,\vec{k}}(\vec{r}) = e^{ik_z z} e^{i\vec{k}_{||} \cdot \vec{\rho}} \vec{u}_{n,\vec{k}_{||}}(\vec{\rho})$$

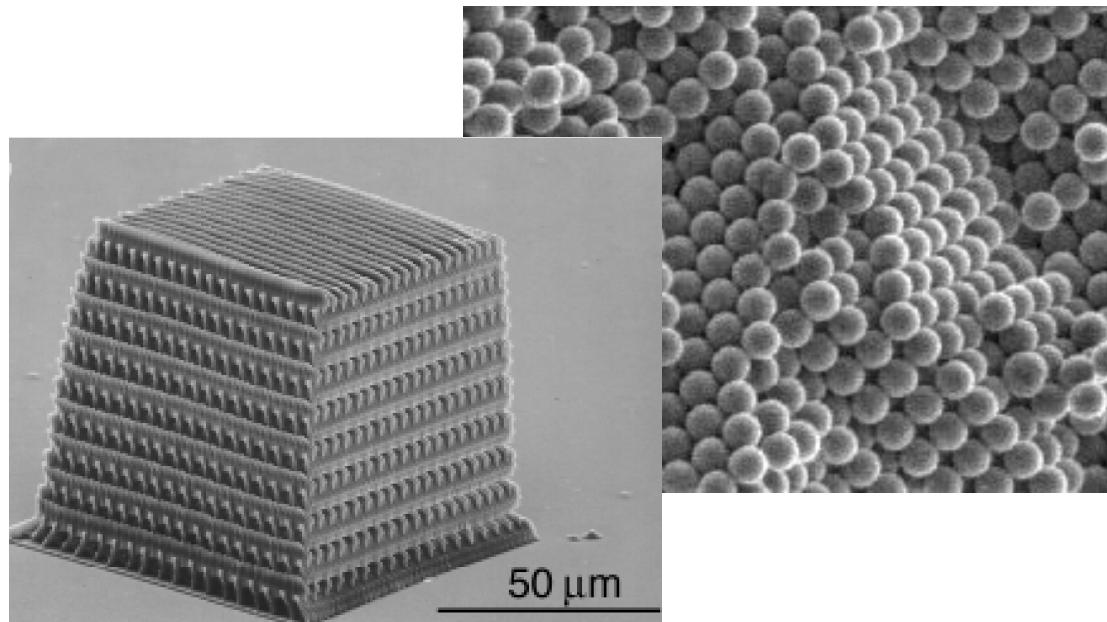
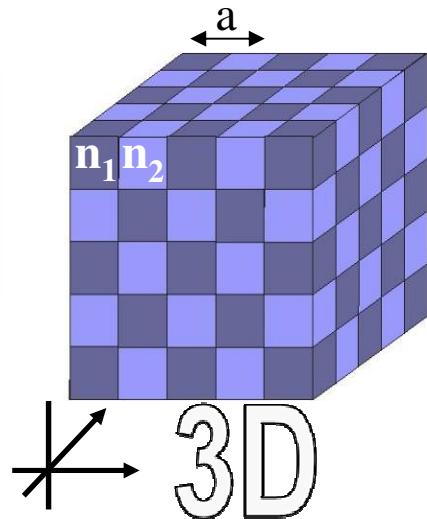
$$\vec{k} = \vec{k}_{||} + \vec{k}_z \quad \vec{\rho} = x\hat{x} + y\hat{y}$$



Γ -point $k=0$
 X -point $k=(\pi/a, 0)$, etc.
 M -point $k=(\pi/a, \pi/a)$, etc.

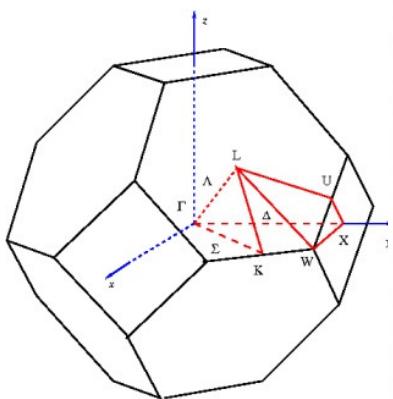


PhC in 3D



$$\vec{H}_{n,\vec{k}}(\vec{r}) = e^{i\vec{k} \cdot \vec{r}} \vec{u}_{n,\vec{k}}(\vec{r})$$

FBZ



Γ -point: $\mathbf{k} = 0$

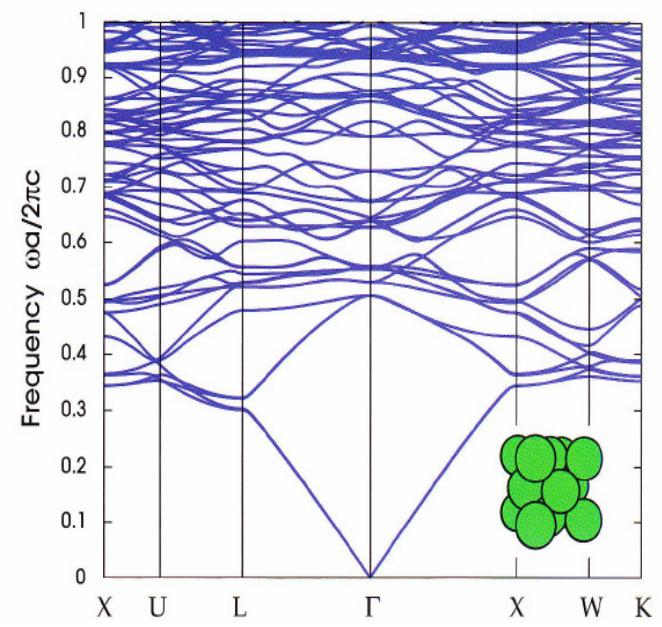
X -points: $\mathbf{k} = \pm \frac{2\pi}{a} \hat{x}, \pm \frac{2\pi}{a} \hat{y}$, etc.

L -points: $\mathbf{k} = \pm \frac{\pi}{a} (\hat{x} + \hat{y} + \hat{z}), \dots$

$\pm \frac{\pi}{a} (\hat{x} - \hat{y} + \hat{z}),$ etc.

K -points: $\mathbf{k} = \pm \frac{3\pi}{2} \frac{\pi}{a} (\hat{x} + \hat{y}),$ etc.

W -points: $\mathbf{k} = \pm \frac{\pi}{a} (2\hat{x} + \hat{y}),$ etc.



Fotonica

Campo

$$\vec{H}(\vec{r},t) = \vec{H}(\vec{r})e^{-i\omega t}$$

Problema
autovalori

$$\hat{\Theta}\vec{H}(\vec{r}) = \frac{\omega^2}{c^2}\vec{H}(\vec{r})$$

Operatore
Hermitiano

$$\hat{\Theta} = \vec{\nabla} \times \frac{1}{\epsilon(\vec{r})} \vec{\nabla} \times$$

Principio
variazionale

$$U_f(\vec{H}) \equiv \frac{(\vec{H}, \hat{\Theta}\vec{H})}{(\vec{H}, \vec{H})}$$

Elettronica

$$\psi(\vec{r},t) = \psi(\vec{r})e^{-i\frac{E}{\hbar}t}$$

$$\hat{H}\psi = E\psi$$

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + V(\vec{r})$$

$$\langle \hat{H} \rangle \equiv \frac{(\psi, \hat{H}\psi)}{(\psi, \psi)}$$

Fotonica

Energia

$$U_H(\vec{H}) \equiv (\mu_0 \vec{H}, \vec{H})$$
$$\vec{S} \equiv (u_E + u_H) \vec{v}_g$$

Legge di
scala

$$\epsilon'(\vec{r}) = \epsilon(\vec{r}/s)$$

$$\omega' = \frac{\omega}{sc}$$

Teorema
Bloch

$$\epsilon(\vec{r} + \vec{R}) = \epsilon(\vec{r})$$

$$\vec{H}(\vec{r}) \equiv \vec{H}_{n,\vec{k}}(\vec{r}) = e^{i\vec{k}\cdot\vec{r}} \vec{u}_{n,\vec{k}}(\vec{r})$$

Elettronica

$$\langle \hat{H} \rangle \equiv \frac{(\psi, \hat{H} \psi)}{(\psi, \psi)}$$

$$a_0 = \frac{4\pi\epsilon_0 \hbar^2}{m_e c^2}$$

$$V(\vec{r} + \vec{R}) = V(\vec{r})$$

$$\psi(\vec{r} + \vec{R}) = e^{i\vec{k}\cdot\vec{r}} u_{n,\vec{k}}(\vec{r})$$

Fotonica

Bande

$$\omega = \omega_n(\vec{k})$$

FBZ

Contiene tutti e soli i
valori non ridondanti di k

Correzione
perturbativa

$$\Delta\omega_n = -\frac{\omega_n}{2} \frac{\int d^3r \vec{E}_n^*(\vec{r}) \delta\epsilon(\vec{r}) \vec{E}_n(\vec{r})}{\int d^3r \vec{E}_n^*(\vec{r}) \epsilon(\vec{r}) \vec{E}_n(\vec{r})}$$

Elettronica

$$E = E_n(\vec{k})$$

Contiene tutti e soli i
valori non ridondanti di k

$$\Delta E_n = \frac{\int d^3r \psi_n^*(\vec{r}) \delta V(\vec{r}) \psi_n(\vec{r})}{\int d^3r \psi_n^*(\vec{r}) \psi_n(\vec{r})}$$