Sommario

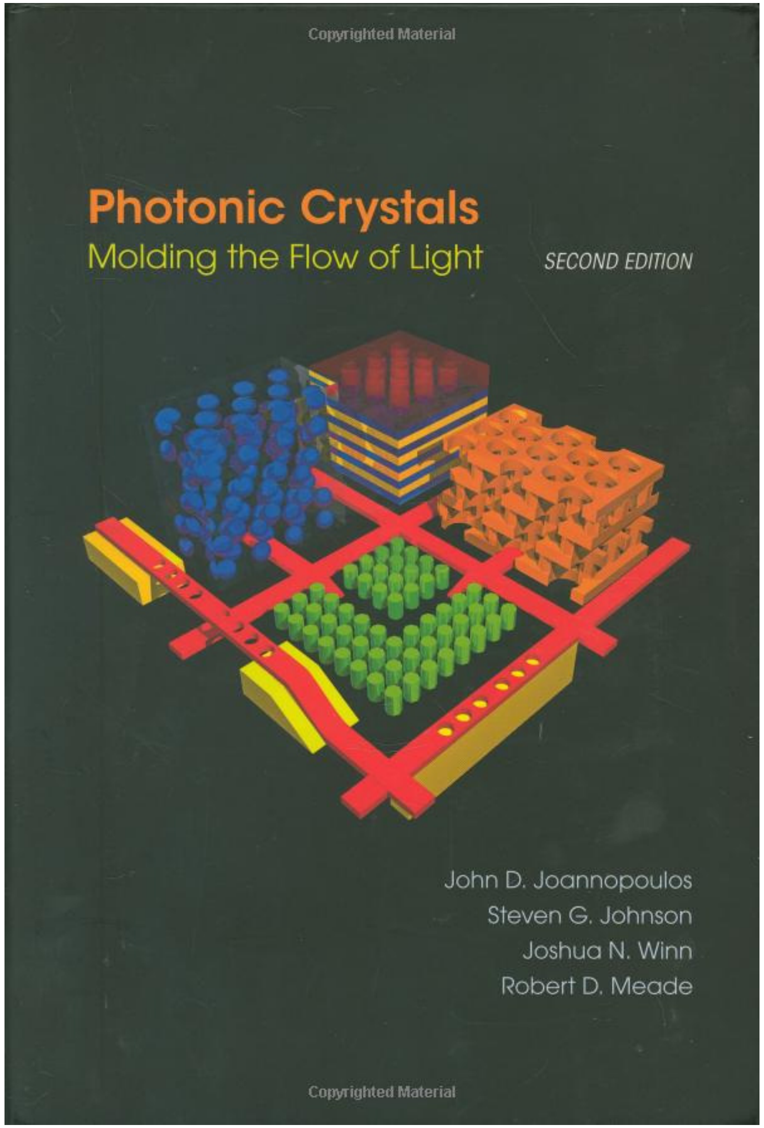
$$\hat{\Theta}\vec{H}(\vec{r}) = \frac{\omega^2}{c^2}\vec{H}(\vec{r})$$ *Master equation*

$$\hat{\Theta}\vec{H}(\vec{r}) = \vec{\nabla}\times\left(\frac{1}{\varepsilon(\vec{r})}\vec{\nabla}\times\vec{H}(\vec{r})\right)$$ *Operatore Hermitiano*

$$\vec{\nabla}\cdot\vec{H}(\vec{r}) = 0$$ *Vincolo onde TM*

$$\omega$$ *Frequenza reale e positiva*

$$(\vec{H}_1, \vec{H}_2) = 0$$ *Ortogonalità autovettori*



http://ab-initio.mit.edu/book/photonic-crystals-book.pdf

Relazioni autovettori

Noto $\left(\vec{H}_2(\vec{r}), \vec{H}_1(\vec{r})\right)$ quanto vale $\left(\vec{E}_2(\vec{r}), \vec{E}_1(\vec{r})\right)$

$$\vec{H}_i(\vec{r}) = \frac{1}{i\omega\mu_0}\vec{\nabla}\times\vec{E}_i(\vec{r}) \qquad i = 1,2$$

$$\left(\vec{H}_2(\vec{r}), \vec{H}_1(\vec{r})\right) = \frac{1}{(\omega\mu_0)^2}\left(\vec{\nabla}\times\vec{E}_2(\vec{r}), \vec{\nabla}\times\vec{E}_1(\vec{r})\right)$$

Essendo

$$\vec{\nabla}\cdot\left[\vec{A}\times\vec{B}\right] = \vec{B}\cdot\vec{\nabla}\times\vec{A} - \vec{A}\cdot\vec{\nabla}\times\vec{B}$$

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{E}\right) = \frac{\omega^2}{c^2}\varepsilon(r)\vec{E}$$

Quindi

$$\left(\vec{H}_1(\vec{r}), \vec{H}_2(\vec{r})\right) = \frac{1}{(\mu_0 c)^2}\left(\vec{E}_1(\vec{r}), \varepsilon(\vec{r})\vec{E}_2(\vec{r})\right)$$

Principio variazionale

$$\boxed{U_f(\vec{H}) \equiv \frac{\left(\vec{H}, \hat{\Theta}\vec{H}\right)}{\left(\vec{H}, \vec{H}\right)} \qquad \text{funzionale energia}}$$

$$U_f(\vec{H} + \delta\vec{H}) = \frac{\left(\vec{H} + \delta\vec{H}, \hat{\Theta}\vec{H} + \hat{\Theta}\,\delta\vec{H}\right)}{\left(\vec{H} + \delta\vec{H}, \vec{H} + \delta\vec{H}\right)} =$$

$$= \frac{\left(\vec{H}, \hat{\Theta}\vec{H}\right) + \left(\vec{H}, \hat{\Theta}\,\delta\vec{H}\right) + \left(\delta\vec{H}, \hat{\Theta}\vec{H}\right)}{\left(\vec{H}, \vec{H}\right) + \left(\vec{H}, \delta\vec{H}\right) + \left(\delta\vec{H}, \vec{H}\right)} =$$

$$= \frac{\left(\vec{H}, \hat{\Theta}\vec{H}\right) + \left(\vec{H}, \hat{\Theta}\,\delta\vec{H}\right) + \left(\delta\vec{H}, \hat{\Theta}\vec{H}\right)}{\left(\vec{H}, \vec{H}\right)} \left(1 - \frac{\left(\vec{H}, \delta\vec{H}\right) + \left(\delta\vec{H}, \vec{H}\right)}{\left(\vec{H}, \vec{H}\right)}\right)$$

Principio variazionale

$$U_f(\vec{H}+\delta\vec{H}) = \frac{\left(\vec{H},\hat{\Theta}\vec{H}\right)}{\left(\vec{H},\vec{H}\right)}\left(1-\frac{\left(\vec{H},\delta\vec{H}\right)+\left(\delta\vec{H},\vec{H}\right)}{\left(\vec{H},\vec{H}\right)}\right)+\frac{\left(\vec{H},\hat{\Theta}\,\delta\vec{H}\right)+\left(\delta\vec{H},\hat{\Theta}\vec{H}\right)}{\left(\vec{H},\vec{H}\right)}=$$

$$=U_f(\vec{H})-\frac{\left(\vec{H},\delta\vec{H}\right)\left(\vec{H},\hat{\Theta}\vec{H}\right)+\left(\delta\vec{H},\vec{H}\right)\left(\vec{H},\hat{\Theta}\vec{H}\right)}{\left(\vec{H},\vec{H}\right)\left(\vec{H},\vec{H}\right)}+\frac{\left(\vec{H},\hat{\Theta}\,\delta\vec{H}\right)+\left(\delta\vec{H},\hat{\Theta}\vec{H}\right)}{\left(\vec{H},\vec{H}\right)}=$$

$$=U_f(\vec{H})-\frac{\left(\vec{H},\delta\vec{H}\right)\left(\vec{H},\hat{\Theta}\vec{H}\right)+\left(\delta\vec{H},\vec{H}\right)\left(\vec{H},\hat{\Theta}\vec{H}\right)}{\left(\vec{H},\vec{H}\right)\left(\vec{H},\vec{H}\right)}+\frac{\left(\hat{\Theta}\vec{H},\delta\vec{H}\right)+\left(\delta\vec{H},\hat{\Theta}\vec{H}\right)}{\left(\vec{H},\vec{H}\right)}$$

$$=U_f(\vec{H})+\frac{1}{\left(\vec{H},\vec{H}\right)}\left[\left(\delta\vec{H},\hat{\Theta}\vec{H}-A\vec{H}\right)+\left(\hat{\Theta}\vec{H}-A\vec{H},\delta\vec{H}\right)\right]$$

$$A=\frac{\left(\vec{H},\hat{\Theta}\vec{H}\right)}{\left(\vec{H},\vec{H}\right)\left(\vec{H},\vec{H}\right)}$$

(si usa $\left(\vec{H},\hat{\Theta}\,\delta\vec{H}\right)=\left(\hat{\Theta}\vec{H},\delta\vec{H}\right)$)

Principio variazionale $\left(\vec{H}, \hat{\Theta}\delta\vec{H}\right) = \left(\hat{\Theta}\vec{H}, \delta\vec{H}\right)$

$$\delta U_f(\vec{H}) = U_f(\vec{H} + \delta\vec{H}) - U_f(\vec{H}) =$$

$$= \frac{1}{2}\left[\left(\delta\vec{H}, \vec{G}(\vec{H})\right) + \left(\vec{G}(\vec{H}), \delta\vec{H}\right)\right]$$

$$\vec{G}(\vec{H}) \equiv \frac{2}{\left(\vec{H}, \vec{H}\right)}\left(\hat{\Theta}\vec{H} - \frac{\left(\vec{H}, \hat{\Theta}\vec{H}\right)}{\left(\vec{H}, \vec{H}\right)}\vec{H}\right)$$

Ricordando che

$$\delta f(\vec{r}) = \frac{\delta\vec{r} \cdot \vec{\nabla} f + \vec{\nabla} f \cdot \delta\vec{r}}{2}$$

$\vec{G}(\vec{H})$ è il rate di variazione del funzionale energia rispetto ad H

Principio variazionale

$$\forall \text{ autostato } \vec{H}_i \qquad \hat{\Theta}\vec{H}_i = \frac{\omega_i^2}{c^2}\vec{H}_i$$

quindi

$$\vec{G}(\vec{H}_i) \equiv \frac{2}{(\vec{H}_i,\vec{H}_i)}\left(\hat{\Theta}\vec{H}_i - \frac{(\vec{H},\hat{\Theta}\vec{H}_i)}{(\vec{H}_i,\vec{H})}\vec{H}_i\right) = 0$$

Principio variazionale:

gli autostati minimizzano il funzionale energia

$$\delta U_f(\vec{H}) = 0 \qquad \forall\, \delta\vec{H}$$

Funzionale energia per gli autostati

$$U_f(\vec{H}) \equiv \frac{\left(\vec{H}, \vec{\nabla}\times\left(\frac{1}{\varepsilon(\vec{r})}\vec{\nabla}\times\vec{H}\right)\right)}{\left(\vec{H},\vec{H}\right)}$$

$$\vec{\nabla}\times\vec{H} = -i\varepsilon_0\varepsilon(\vec{r})\omega\vec{E} \qquad \vec{H} = -\frac{i}{\mu_0\omega}\vec{\nabla}\times\vec{E}$$

$$\left(\vec{H}, \vec{\nabla}\times\left(\frac{1}{\varepsilon(\vec{r})}\vec{\nabla}\times\vec{H}\right)\right) = \left(-\frac{i}{\mu_0\omega}\vec{\nabla}\times\vec{E}, \vec{\nabla}\times\left(-i\varepsilon_0\omega\vec{E}\right)\right)$$

$$\left(\vec{H},\vec{H}\right) = \frac{1}{(\mu_0 c)^2}\left(\vec{E},\varepsilon(\vec{r})\vec{E}\right)$$ Vedi passaggi relazioni autovettori

$$U_f(\vec{H}) = \frac{\int d^3r\left|\vec{\nabla}\times\vec{E}\right|^2}{\int d^3r\,\varepsilon(\vec{r})\left|\vec{E}\right|^2}$$

Minimizzare i nodi
Massimizzare campo nel dielettrico

Principio variazionale

Principio variazionale:
gli autostati minimizzano il funzionale energia, quindi i modi fotonici di più bassa frequenza

$$\boxed{U_f(\vec{H}) \equiv \frac{\int d^3r \left|\vec{\nabla}\times\vec{E}\right|^2}{\int d^3r\,\varepsilon(\vec{r})\left|\vec{E}\right|^2}}$$

hanno ampiezza concentrata nella regione ad alto dielettrico. Inoltre un dato modo in generale conterrà più nodi rispetto a un modo di minore frequenza.

In (MQ) le funzioni d'onda di più bassa energia hanno ampiezza concentrata nelle regioni a potenziale minore. Vale anche in MQ la "legge dei nodi".

$$\boxed{\left\langle \hat{H} \right\rangle \equiv \frac{\int d^3r\,\psi^* \hat{H}\varphi}{\int d^3r \left|\psi\right|^2}}$$

Energia elettromagnetica

$$U_E = \frac{1}{2}\left(\varepsilon_o \varepsilon(\vec{r})\vec{E}(\vec{r}), \vec{E}(\vec{r})\right) = \frac{\varepsilon_o}{2}\int d^3 r \varepsilon(\vec{r})\left|\vec{E}(\vec{r})\right|^2 = \int d^3 r u_E$$

$$U_H = \frac{1}{2}\left(\mu_o \vec{H}(\vec{r}), \vec{H}(\vec{r})\right) = \frac{\mu_o}{2}\int d^3 r \left|\vec{H}(\vec{r})\right|^2 = \int d^3 r u_H$$

Continuo scambio fra E e H

$$U_H = \left(\mu_o \vec{H}(\vec{r}), \vec{H}(\vec{r})\right) = \left(\mu_o \vec{H}(\vec{r}), \frac{1}{i\omega\mu_o}\vec{\nabla}\times\vec{E}(\vec{r})\right) =$$

$$= \left(\vec{\nabla}\times\vec{H}(\vec{r}), \frac{1}{i\omega}\vec{E}(\vec{r})\right) = \left(\frac{1}{i\omega} i\omega\varepsilon_o\varepsilon(\vec{r})\vec{E}(\vec{r}), \vec{E}(\vec{r})\right) =$$

$$= \frac{1}{2}\left(\varepsilon_o \varepsilon(\vec{r})\vec{E}(\vec{r}), \vec{E}(\vec{r})\right) = U_E$$

$$\vec{S}(\vec{r},t) = \vec{E}(\vec{r},t) \times \vec{H}(\vec{r},t)$$ Vettore di Poynting

$$\vec{S}(\vec{r},t) = \frac{1}{4}\left(\vec{E}(\vec{r},t) + \vec{E}^*(\vec{r},t)\right) \times \left(\vec{H}(\vec{r},t) + \vec{H}^*(\vec{r},t)\right)$$

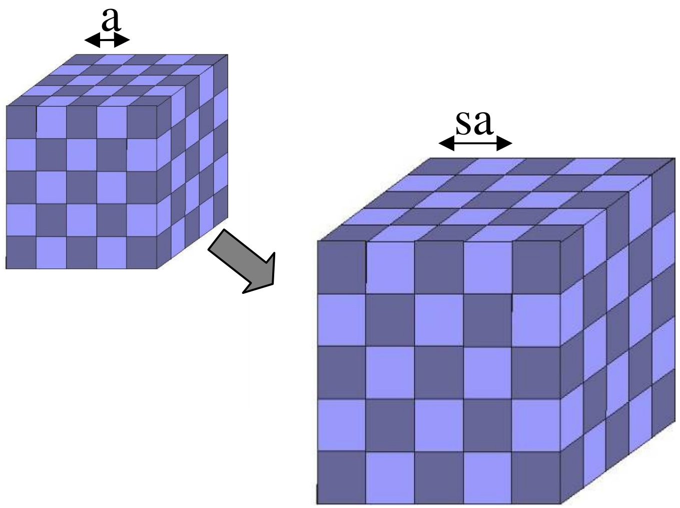
$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{4}\left(\left\langle \vec{E}(\vec{r},t) \times \vec{H}^*(\vec{r},t) \right\rangle + \left\langle \vec{E}^*(\vec{r},t) \times \vec{H}(\vec{r},t) \right\rangle\right)$$

$$\left\langle \vec{S}(\vec{r},t) \right\rangle = \frac{1}{2}\Re e\left[\left\langle \vec{E}(\vec{r},t) \times \vec{H}^*(\vec{r},t) \right\rangle\right]$$

$$\boxed{\vec{S} \equiv \left(u_E + u_H\right)\vec{v}_e \qquad \vec{v}_e = \vec{v}_g = \vec{\nabla}_{\vec{k}}\,\omega(\vec{k})}$$

Proprietà di scala 1

$$\vec{\nabla}\times\left(\frac{1}{\varepsilon(\vec{r})}\vec{\nabla}\times\vec{H}(\vec{r})\right)=\frac{\omega^2}{c^2}\vec{H}(\vec{r})$$



Supponiamo

$$\varepsilon^{'}(\vec{r})=\varepsilon(\vec{r}/s)$$

$$\vec{r}^{'}=s\vec{r} \qquad \vec{\nabla}^{'}=\vec{\nabla}/s$$

Ogni effetto è identico nelle micronde, nell'IR, in ottica o in raggi X

Sostituendo

$$\vec{\nabla}^{'}\times\left(\frac{1}{\varepsilon^{'}(\vec{r}^{'})}\vec{\nabla}^{'}\times\vec{H}(\vec{r}^{'}/s)\right)=\frac{\omega^2}{s^2c^2}\vec{H}(\vec{r}^{'}/s)$$

quindi

$$\vec{H}^{'}(\vec{r}^{'})=\vec{H}(\vec{r}^{'}/s) \qquad \omega^{'}=\frac{\omega}{s}$$

Proprietà di scala 2

$$\vec{\nabla}\times\left(\frac{1}{\varepsilon(\vec{r})}\vec{\nabla}\times\vec{H}(\vec{r})\right)=\frac{\omega^2}{c^2}\vec{H}(\vec{r})$$

Supponiamo

$$\varepsilon^{'}(\vec{r})=\varepsilon(\vec{r})/s^2$$

Sostituendo

$$\vec{\nabla}^{'}\times\left(\frac{1}{\varepsilon^{'}(\vec{r})}\vec{\nabla}^{'}\times\vec{H}(\vec{r}^{\,'})\right)=\frac{(s\omega)^2}{s^2c^2}\vec{H}(\vec{r}^{\,'})$$

$$\vec{H}^{'}=\vec{H} \qquad\qquad \omega^{'}=s\omega$$

Gli autovettori (i modi) restano invariati ma le frequenze sono scalate di s

Spettro discreto vs continuo

$$\vec{H}_{\omega}(\vec{r}) \quad \omega \textit{ ha spettro continuo}$$

$$\left(\vec{H}_{\omega}(\vec{r}), \vec{H}_{\omega+\delta\omega}(\vec{r})\right) = 0 \quad \forall \delta\omega$$

$$\vec{H}_{\omega+\delta\omega}(\vec{r}) = \vec{H}_{\omega}(\vec{r}) + \delta\vec{H}$$

$$\left(\vec{H}_{\omega}(\vec{r}), \vec{H}_{\omega+\delta\omega}(\vec{r})\right) = \left(\vec{H}_{\omega}(\vec{r}), \vec{H}_{\omega}(\vec{r}) + \delta\vec{H}(\vec{r})\right) =$$
$$\left(\vec{H}_{\omega}(\vec{r}), \vec{H}_{\omega}(\vec{r})\right) + \left(\vec{H}_{\omega}(\vec{r}), \delta\vec{H}(\vec{r})\right)$$

Se e solo se $V = \infty$

$$\left(\vec{H}_{\omega}(\vec{r}), \delta\vec{H}(\vec{r})\right) = \int_V d^3 r \vec{H}_{\omega}^{\,*}(\vec{r}) \cdot \delta\vec{H}(\vec{r}) \not\approx \mathrm{O}(\delta\omega)$$

Stati localizzati $\rightarrow$ spettro discreto

Stati delocalizzati $\rightarrow$ spettro continuo