

The Wave Equation

$$\nabla^2 \vec{E} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} = \mu_0 \frac{\partial^2 \vec{P}}{\partial t^2}$$

The Polarization

$$\vec{P} = N \langle \Psi | e \vec{r} | \Psi \rangle \quad \tilde{\rho} = |\Psi\rangle \langle \Psi|$$

$$\vec{P} = N \sum_n \sum_m \tilde{\rho}_{nm} \langle n | e \vec{r} | m \rangle \quad \tilde{\rho}_{nm} = \langle n | \tilde{\rho} | m \rangle$$

The Liouville Equation

$$i\hbar \dot{\tilde{\rho}} = [\hat{H}, \tilde{\rho}] = [(\hat{H}_a + \hat{H}_I), \tilde{\rho}] + i\hbar \hat{\Gamma} \tilde{\rho}$$

The Electric Field

$$E(z,t) = E_a(z,t) \times \exp[i\omega_a(t - z/c)] \\ + E_b(z,t) \times \exp[i\omega_b(t - z/c)] + c.c$$

The Polarization

$$\wp(z,t) = N \langle \Psi | er | \Psi \rangle = \{ \wp_a(z,t) \times \exp[i\omega_a(t - z/c)] + \\ \wp_b(z,t) \times \exp[i\omega_b(t - z/c)] \} + c.c.$$

$$\wp_a(z,t) = N d_{13} \rho_{13}(z,t) \quad ; \quad \wp_b(z,t) = N d_{23} \rho_{23}(z,t)$$

The Propagation Equations

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2} \frac{\partial^2}{\partial t^2} \right) E_m(z,t) \times \exp[i\omega_m(t - z/c)] =$$

$$\mu_0 \frac{\partial^2}{\partial t^2} \wp_m(z,t) \times \exp[i\omega_m(t - z/c)]$$

$$\frac{\partial^2}{\partial z^2} \{E_m(z, t) \times \exp[i\omega_m(t - z/c)]\} =$$

$$\left\{ \frac{\partial^2 E_m}{\partial z^2} - 2i \frac{\omega_m}{c} \frac{\partial E_m}{\partial z} - \frac{\omega_m^2}{c^2} E_m \right\} \times \exp[i\omega_m(t - z/c)]$$

$$\frac{\partial^2}{\partial t^2} \{E_m(z, t) \times \exp[i\omega_m(t - z/c)]\} =$$

$$\left\{ \frac{\partial^2 E_m}{\partial t^2} + 2i\omega_m \frac{\partial E_m}{\partial t} - \omega_m^2 E_m \right\} \times \exp[i\omega_m(t - z/c)]$$

$$\frac{\partial^2}{\partial t^2} \{\phi_m(z, t) \times \exp[i\omega_m(t - z/c)]\} =$$

$$\left\{ \frac{\partial^2 \phi_m}{\partial t^2} + 2i\omega_m \frac{\partial \phi_m}{\partial t} - \omega_m^2 \phi_m \right\} \times \exp[i\omega_m(t - z/c)]$$

The Slow Varying Envelope Approximation (SVEA)

$$\frac{\partial^2 E_m}{\partial t^2} \ll \omega_m \frac{\partial E_m}{\partial t}$$

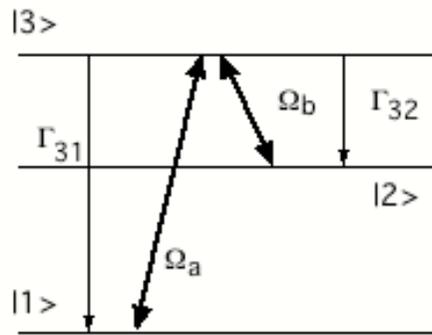
$$\frac{\partial^2 E_m}{\partial z^2} \ll \frac{\omega_m}{c} \frac{\partial E_m}{\partial z}$$

$$\frac{\partial^2 \rho_m}{\partial t^2} \ll \omega_m \frac{\partial \rho_m}{\partial t} \ll \omega_m^2 \rho_m$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) E_a = -iN \frac{\omega_a d_{13}}{2\epsilon_0 c} \rho_{13}$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t} \right) E_b = -iN \frac{\omega_b d_{23}}{2\epsilon_0 c} \rho_{23}$$

Radiatively broadened 3-level Λ system



$$\dot{\rho}_{11} = i\frac{\Omega_a}{2}(\rho_{13}^* - \rho_{13}) + \Gamma_{31}\rho_{33}$$

$$\dot{\rho}_{22} = i\frac{\Omega_b}{2}(\rho_{23}^* - \rho_{23}) + \Gamma_{32}\rho_{33}$$

$$\dot{\rho}_{33} = i\frac{\Omega_a}{2}(\rho_{13} - \rho_{13}^*) + i\frac{\Omega_b}{2}(\rho_{23} - \rho_{23}^*) - (\Gamma_{31} + \Gamma_{32})\rho_{33}$$

$$\dot{\rho}_{12} = i\frac{\Omega_a}{2}\rho_{23}^* - i\frac{\Omega_b}{2}\rho_{13} + [i(\Delta_b - \Delta_a) - \gamma_{12}]\rho_{12}$$

$$\dot{\rho}_{13} = i\frac{\Omega_a}{2}(\rho_{33} - \rho_{11}) - i\frac{\Omega_b}{2}\rho_{12} - (i\Delta_a + \gamma_{13})\rho_{13}$$

$$\dot{\rho}_{23} = i\frac{\Omega_b}{2}(\rho_{33} - \rho_{22}) - i\frac{\Omega_a}{2}\rho_{12} - (i\Delta_b + \gamma_{23})\rho_{23}$$

$$\Omega_a = \frac{E_a d_{13}}{\hbar}$$

$$\Omega_b = \frac{E_b d_{23}}{\hbar}$$

$$\Delta_a = \omega_a - \frac{E_3 - E_1}{\hbar}$$

$$\Delta_b = \omega_b - \frac{E_3 - E_2}{\hbar}$$

- Weak probe field ($\rho_{11} = 1$)
- Resonant coupling field ($\Delta_b = 0$)
- Constant amplitude (C.W.) of coupling field
- Arbitrary amplitude (pulsed) of probe field

$$\dot{\rho}_{12} = -i\frac{\Omega_b}{2}\rho_{13} - (i\Delta_a + \gamma_{12})\rho_{12}$$

$$\dot{\rho}_{13} = -i\frac{\Omega_a}{2} - i\frac{\Omega_b}{2}\rho_{12} - (i\Delta_a + \gamma_{13})\rho_{13}$$

$$E_a(z,t) = \int_{-\infty}^{+\infty} S_a(\omega, z) \exp(i\omega t) d\omega$$

$$\rho_{12}(z,t) = -\frac{d_{13}}{2\hbar} \frac{\Omega_b}{2} \int_{-\infty}^{+\infty} S_a(\omega, z) f_{12}(\omega) \exp(i\omega t) d\omega$$

$$\rho_{13}(z,t) = \frac{d_{13}}{2\hbar} \int_{-\infty}^{+\infty} S_a(\omega, z) f_{13}(\omega) \exp(i\omega t) d\omega$$

$$f_{12}(\omega) = \frac{1}{[i(\omega + \Delta_a) + \gamma_{12}][i(\omega + \Delta_a) + \gamma_{13}] + (\Omega_b/2)^2}$$

$$f_{13}(\omega) = \frac{(\omega + \Delta_a - i\gamma_{12})}{[i(\omega + \Delta_a) + \gamma_{12}][i(\omega + \Delta_a) + \gamma_{13}] + (\Omega_b/2)^2}$$

The propagation equation for $S_a(\omega, z)$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c} \frac{\partial}{\partial t}\right) E_a = -iN \frac{\omega_a d_{13}}{2\epsilon_0 c} \rho_{13}$$

$$E_a(z, t) = \int_{-\infty}^{+\infty} S_a(\omega, z) \exp(i\omega t) d\omega$$

$$\rho_{13}(z, t) = \frac{d_{13}}{2\hbar} \int_{-\infty}^{+\infty} S_a(\omega, z) f_{13}(\omega) \exp(i\omega t) d\omega$$

$$\frac{\partial}{\partial z} S_a(\omega, z) = -i\beta(\omega) S_a(\omega, z)$$

$$\beta(\omega) = \frac{\omega}{c} + N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} f_{13}(\omega)$$

$$E_a(z, t) = \int_{-\infty}^{+\infty} S_a(\omega, 0) \exp\{i[\omega t - \beta(\omega)z]\} d\omega$$

Two-level atom ($\Omega_b = 0$)

$$f_{13}(\omega) = \frac{1}{[i\gamma_{13} - (\omega + \Delta_a)]}$$

$$\delta\omega \ll \sqrt{\Delta_a^2 + \gamma_{13}^2}$$

$$f_{13}(\omega) \approx f_{13}(0) + f'_{13}(0)\omega$$

$$f_{13}(0) = -\frac{\Delta_a}{\Delta_a^2 + \gamma_{13}^2} - i\frac{\gamma_{13}}{\Delta_a^2 + \gamma_{13}^2}$$

$$f'_{13}(0) = \frac{\Delta_a^2 - \gamma_{13}^2}{(\Delta_a^2 + \gamma_{13}^2)^2} + i\frac{2\Delta_a\gamma_{13}}{(\Delta_a^2 + \gamma_{13}^2)^2}$$

$$\beta(\omega) = \frac{\omega}{c} + N\frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} f_{13}(\omega) \approx$$

$$\approx N\frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \text{Re}[f_{13}(0)] + iN\frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \text{Im}[f_{13}(0)] +$$

$$\left(\frac{1}{c} + N\frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \text{Re}[f'_{13}(0)]\right)\omega$$

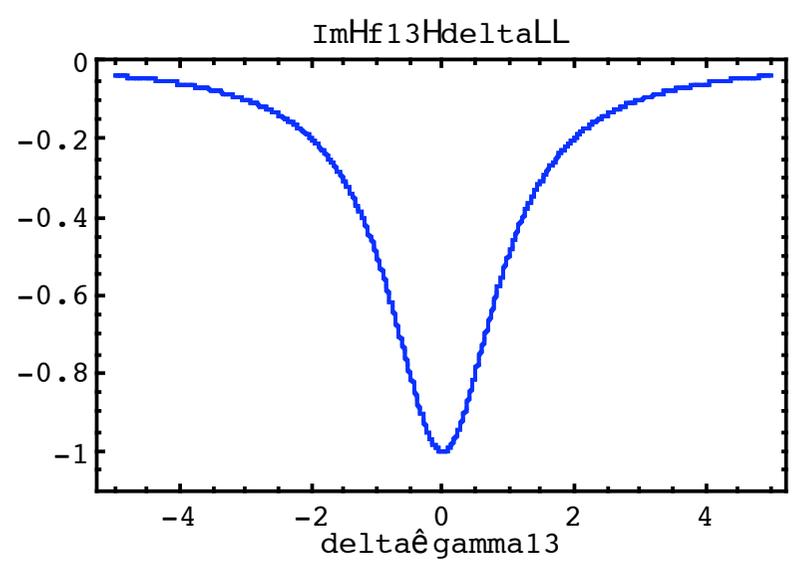
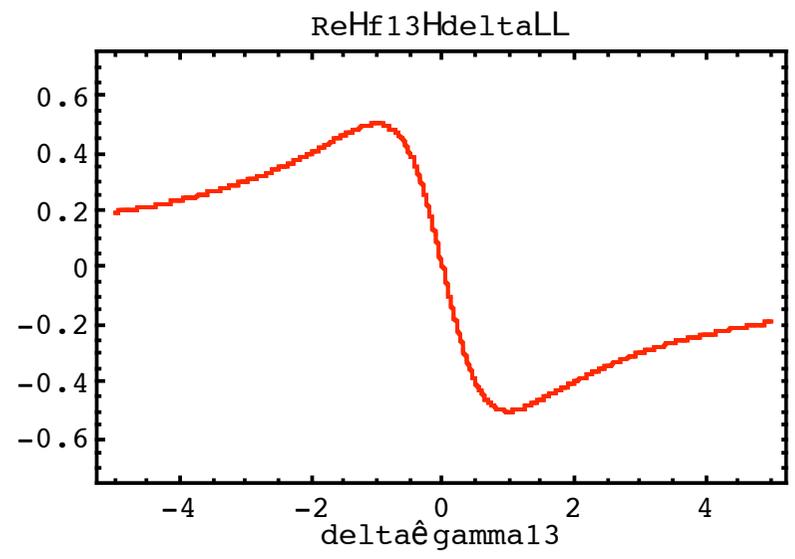
The complete probe field E_a is given by:

$$\begin{aligned} E_a &= E_a(z, t) \times \exp[i\omega_a(t - z/c)] = \\ &= E_a(t - z/u_g) \times \exp[i\omega_a(t - z/u_f) - \alpha_0 z] \end{aligned}$$

$$\frac{1}{u_f} = \frac{1}{c} + N \frac{d_{13}^2}{4\hbar\epsilon_0 c} \text{Re}[f_{13}(0)]$$

$$\alpha_0 = -N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \text{Im}[f_{13}(0)]$$

$$\frac{1}{u_g} = \frac{1}{c} + N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \text{Re}[f'_{13}(0)]$$



$$\text{Re}[f_{13}(0)] = -\frac{\Delta_a}{\Delta_a^2 + \gamma_{13}^2}$$

$$\text{Im}[f_{13}(0)] = -\frac{\gamma_{13}}{\Delta_a^2 + \gamma_{13}^2}$$

$$\text{Re}[f'_{13}(0)] = \frac{\Delta_a^2 - \gamma_{13}^2}{(\Delta_a^2 + \gamma_{13}^2)^2}$$

$$\frac{1}{u_f} = \frac{1}{c} - N \frac{d_{13}^2}{4\hbar\epsilon_0 c} \frac{\Delta_a}{\Delta_a^2 + \gamma_{13}^2}$$

$$\alpha_0 = N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{\gamma_{13}}{\Delta_a^2 + \gamma_{13}^2}$$

$$\frac{1}{u_g} = \frac{1}{c} + N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{\Delta_a^2 - \gamma_{13}^2}{(\Delta_a^2 + \gamma_{13}^2)^2}$$

Line center ($\Delta_a = 0$)

$$\alpha_0 = N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{1}{\gamma_{13}}$$

$$\frac{1}{u_g} = \frac{1}{c} - N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{1}{\gamma_{13}^2}$$

Due livelli: propagazione alla risonanza

Caso propagazione vuoto $N=0$

$$I_a(t,z) = E_a^2(t-z/c)$$

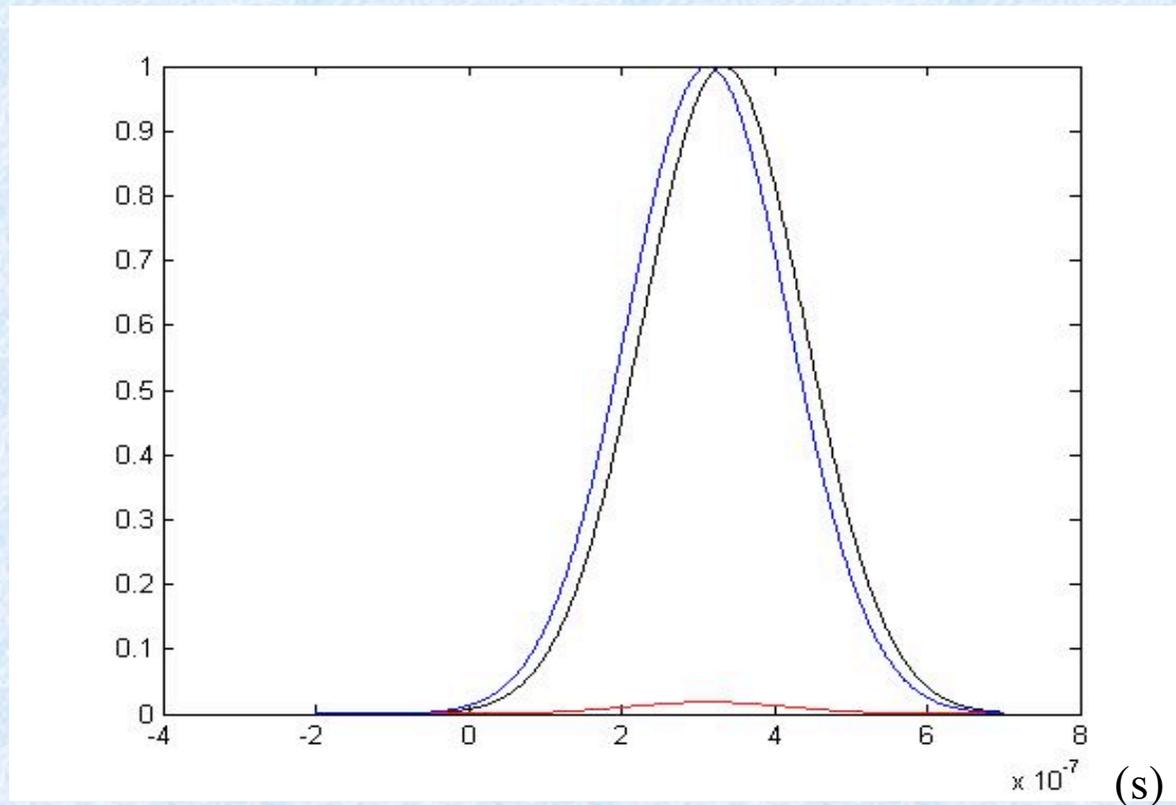
Caso propagazione mezzo materiale

$$I_a(t,z) = E_a^2(t-z/u_g) \exp(-2\alpha_0 z)$$

Caso propagazione mezzo materiale
 $\alpha_0=0$ (irrealistico)

$$I_a(t,z) = E_a^2(t-z/u_g)$$

Andamento temporale



$z=100$ m;
 $\alpha_0=0.02$

Massimo dell'impulso

$I_a(t,z)$ $t_{\max}=333$ ns

$I_a(t,z)$, $I_a(t,z)$
per $t_{\max}=313$ ns

$u_g/c=1.064$

Due livelli: propagazione alla risonanza

Caso propagazione vuoto $N=0$

$$I_a(t,z) = E_a^2(t-z/c)$$

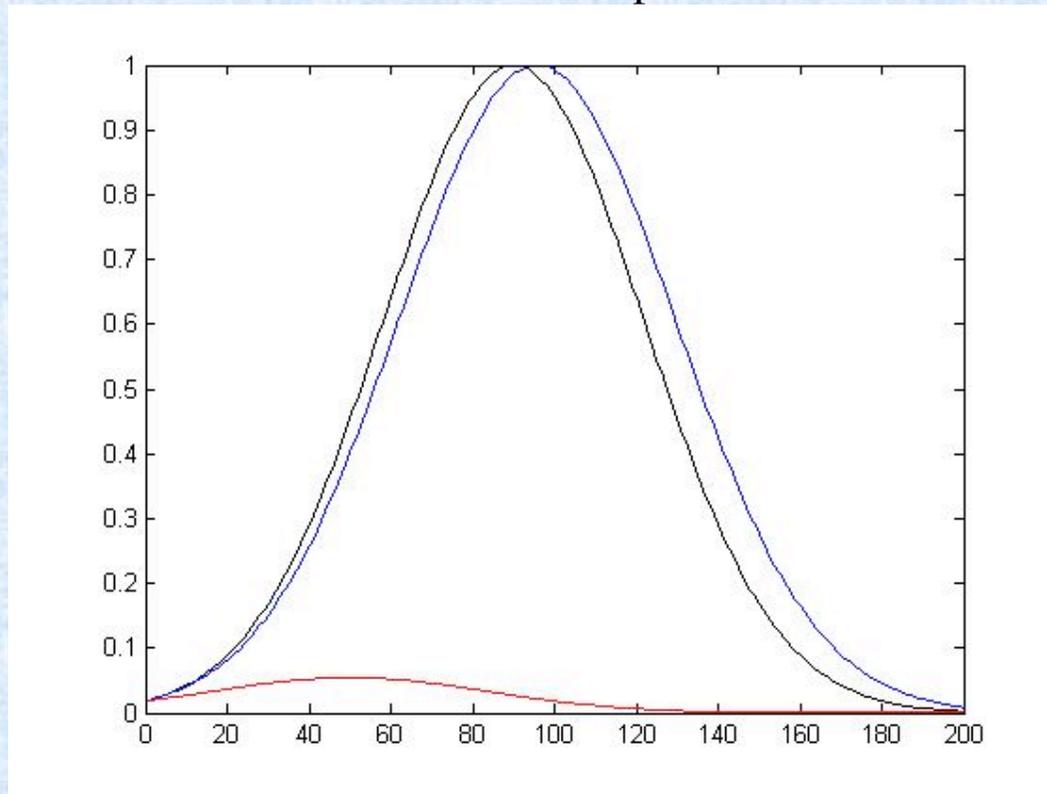
Caso propagazione mezzo materiale

$$I_a(t,z) = E_a^2(t-z/u_g) \exp(-2\alpha_0 z)$$

Caso propagazione mezzo materiale
 $\alpha_0=0$ (irrealistico)

$$I_a(t,z) = E_a^2(t-z/u_g)$$

Andamento spaziale



$$t=300 \text{ ns};$$
$$\alpha_0=0.02$$

Massimo dell'impulso

$$I_a(t,z) \quad z_{\max}=90 \text{ m}$$

$$I_a(t,z) \quad z_{\max}=95.7 \text{ m}$$

$$I_a(t,z) \quad z_{\max}=50 \text{ m}$$

$$u_g/c=1.064$$

Due livelli: propagazione alla risonanza con forte assorbimento

Caso propagazione vuoto $N=0$

$$I_a(t,z) = E_a^2(t-z/c)$$

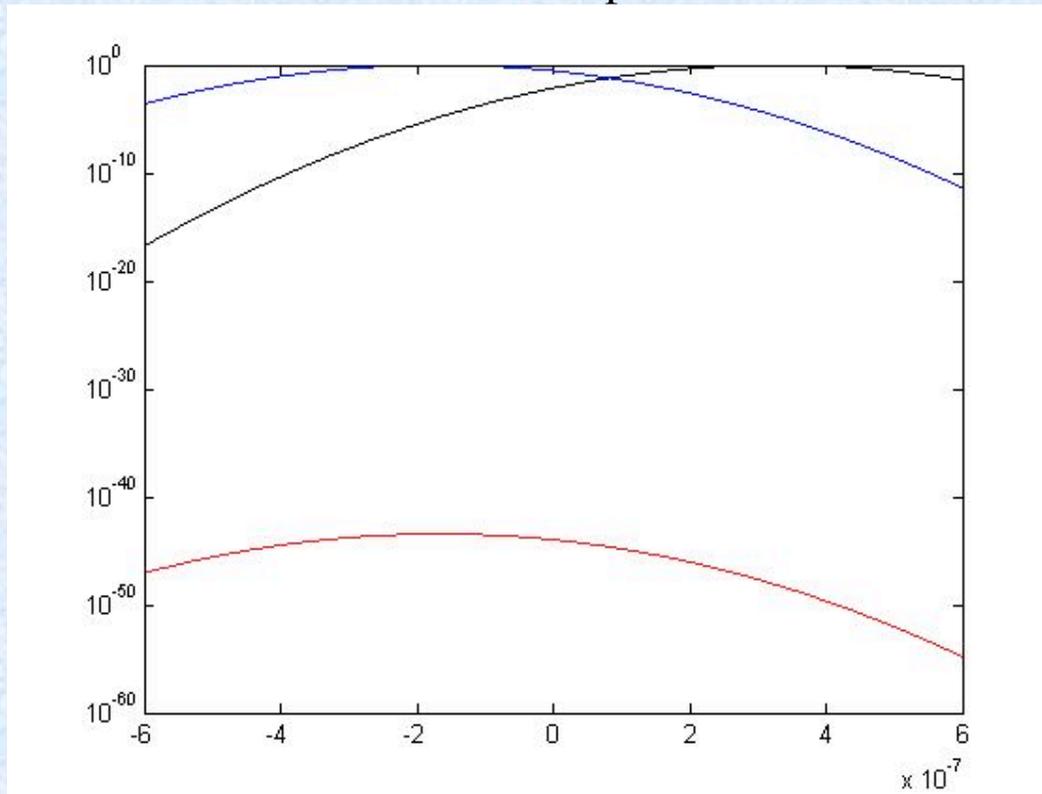
Caso propagazione mezzo materiale

$$I_a(t,z) = E_a^2(t-z/u_g) \exp(-2\alpha_0 z)$$

Caso propagazione mezzo materiale
 $\alpha_0=0$ (irrealistico)

$$I_a(t,z) = E_a^2(t-z/u_g)$$

Andamento temporale



$z=100$ m;
 $\alpha_0=0.5$

Massimo dell'impulso

$I_a(t,z)$ $t_{\max}=333$ ns

$I_a(t,z)$, $I_a(t,z)$
per $t_{\max} = -167$ ns

$u_g/c = -2.0$

Due livelli: propagazione alla risonanza con forte assorbimento

Caso propagazione vuoto $N=0$

$$I_a(t,z)=E_a^2(t-z/c)$$

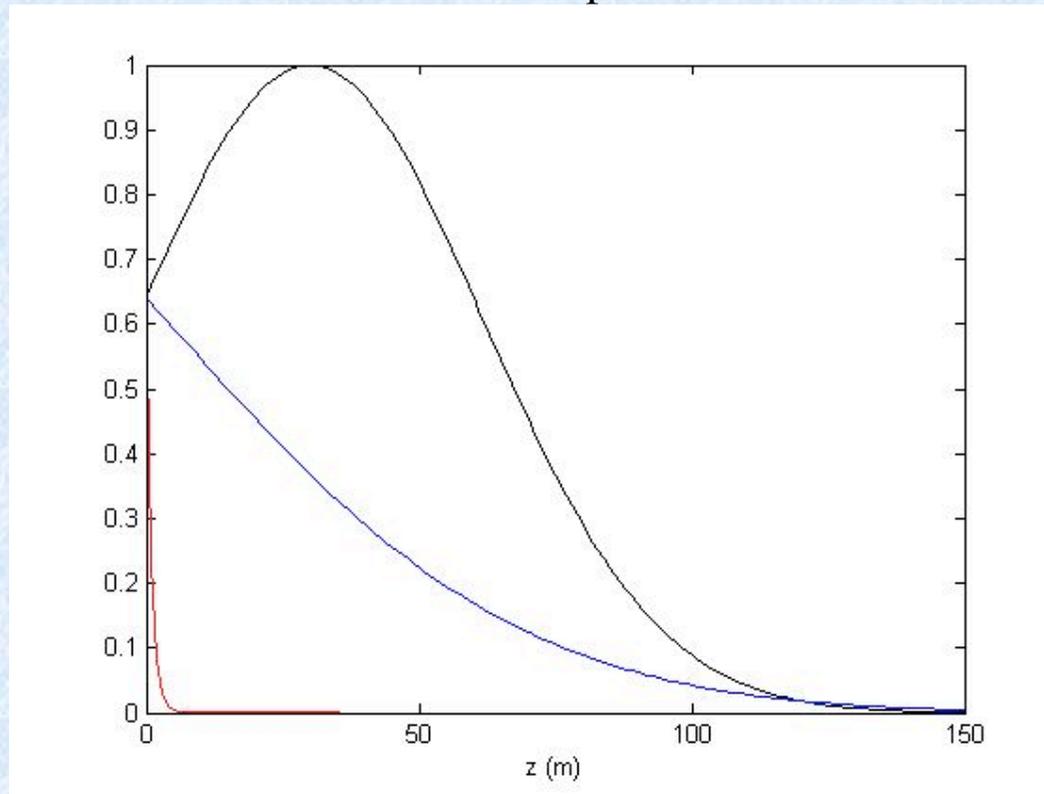
Caso propagazione mezzo materiale

$$I_a(t,z)=E_a^2(t-z/u_g) \exp(-2\alpha_0 z)$$

Caso propagazione mezzo materiale
 $\alpha_0=0$ (irrealistico)

$$I_a(t,z)=E_a^2(t-z/u_g)$$

Andamento spaziale



$$t=100 \text{ ns};$$
$$\alpha_0=0.5$$

$$u_g/c=-2.0$$

Due livelli: propagazione alla risonanza con forte assorbimento

Caso propagazione vuoto $N=0$

$$I_a(t,z) = E_a^2(t-z/c)$$

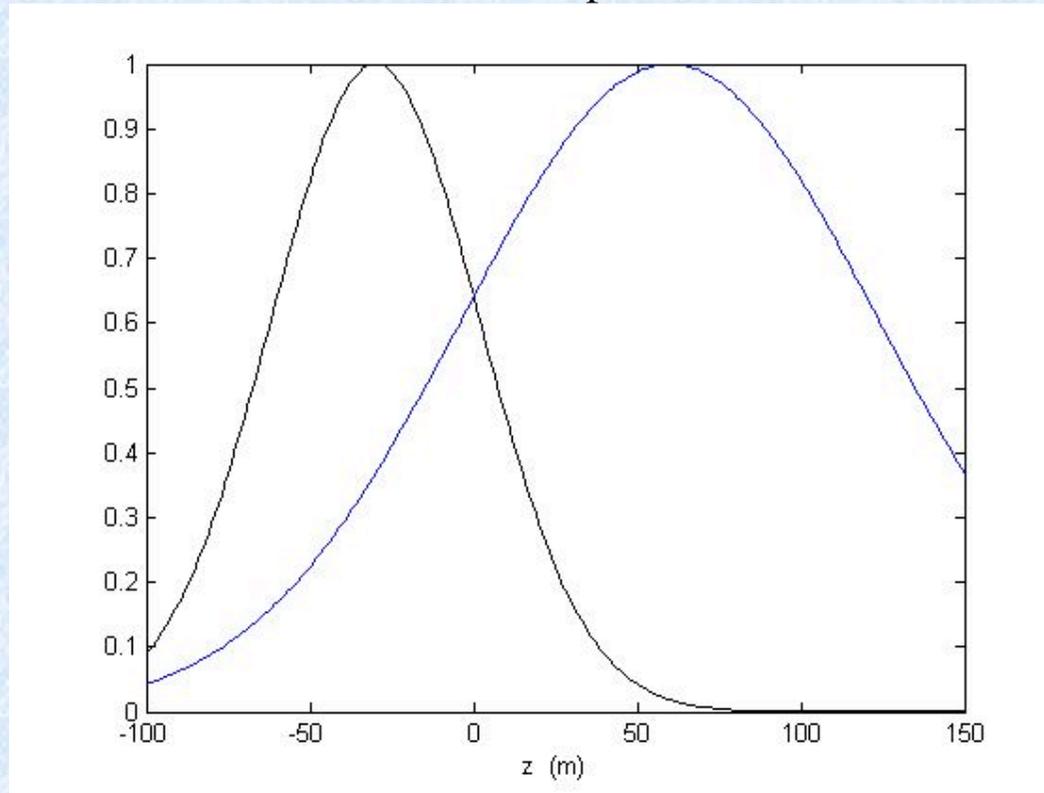
Caso propagazione mezzo materiale

$$I_a(t,z) = E_a^2(t-z/u_g) \exp(-2\alpha_0 z)$$

Caso propagazione mezzo materiale
 $\alpha_0=0$ (irrealistico)

$$I_a(t,z) = E_a^2(t-z/u_g)$$

Andamento spaziale



$t = -100$ ns;

$\alpha_0 = 0.5$

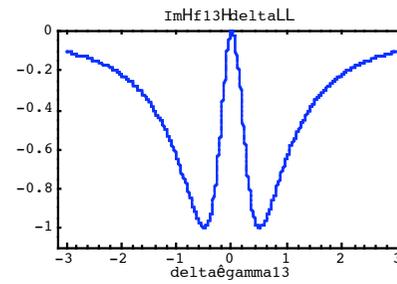
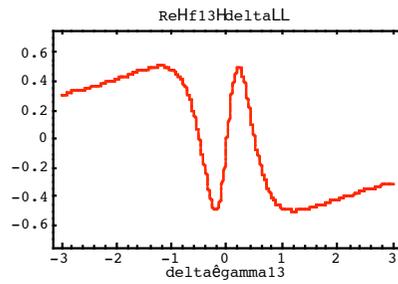
Massimo dell'impulso

$I_a(t,z)$ $z_{\max} = -30$ m

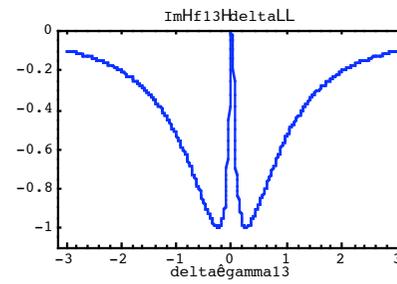
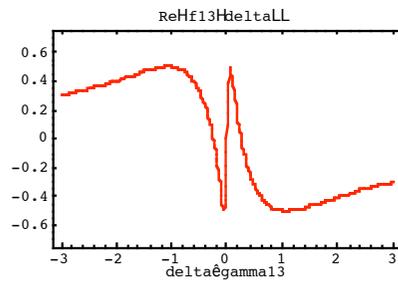
$I_a(t,z)$ $z_{\max} = 60$ m

$u_g/c = -2.0$

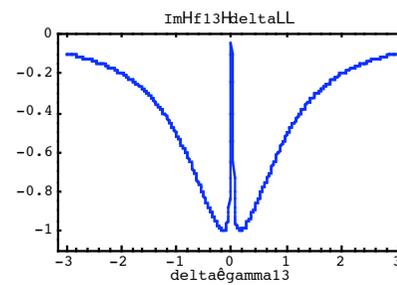
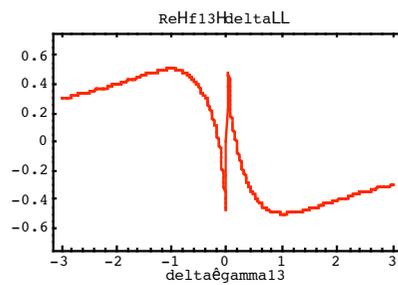
Three-level atom $\gamma_{12} = 0.001\gamma_{13}$



$$\Omega_b = \gamma_{13}$$



$$\Omega_b = 0.5\gamma_{13}$$



$$\Omega_b = 0.3\gamma_{13}$$

Line center ($\Delta_a = 0$)

$$\frac{1}{u_g} = \frac{1}{c} + N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{(\Omega_b/2)^2 - \gamma_{12}^2}{[\gamma_{12}\gamma_{13} + (\Omega_b/2)^2]^2}$$

$$\alpha_a(0) = N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{\gamma_{12}}{\gamma_{12}\gamma_{13} + (\Omega_b/2)^2}$$

$$(\Omega_b/2)^2 \gg \gamma_{12}\gamma_{13}, \gamma_{12}^2$$

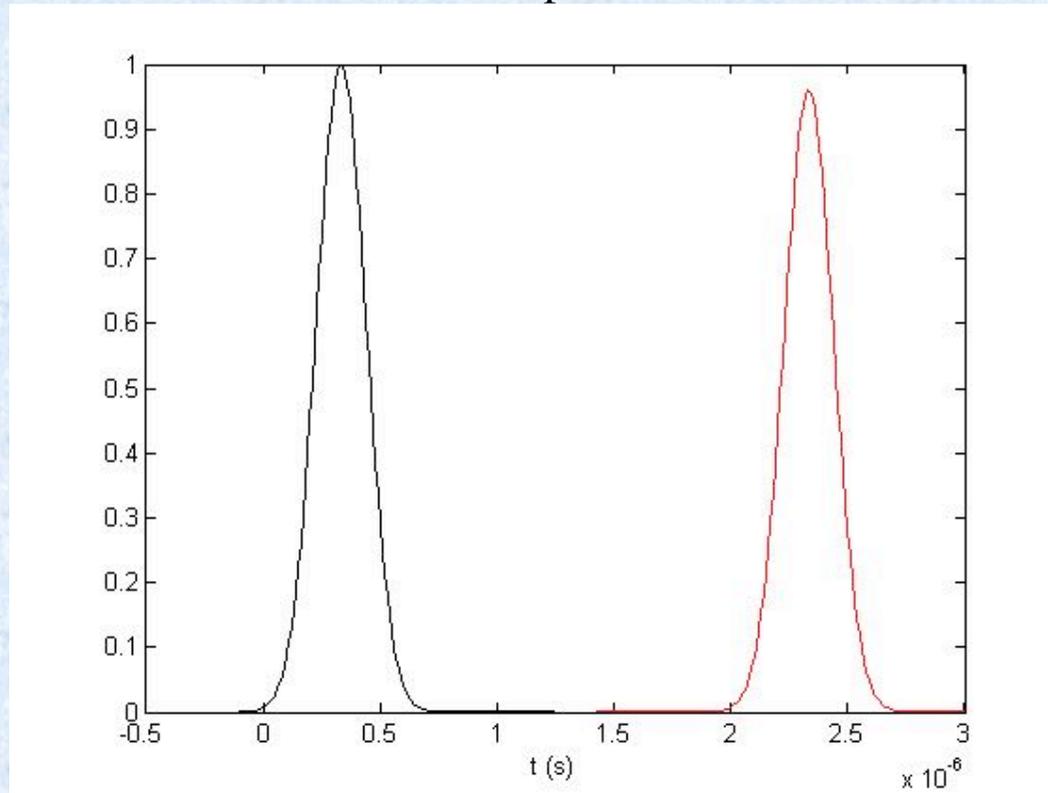
$$\frac{1}{u_g} = \frac{1}{c} + N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{1}{(\Omega_b/2)^2} ; \quad \alpha_a(0) = N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{\gamma_{12}}{(\Omega_b/2)^2}$$

$$(\Omega_b/2)^2 \ll \gamma_{12}\gamma_{13}, \gamma_{12}^2$$

$$\frac{1}{u_g} = \frac{1}{c} - N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{1}{\gamma_{13}^2} ; \quad \alpha_a(0) = N \frac{\omega_a d_{13}^2}{4\hbar\epsilon_0 c} \frac{1}{\gamma_{13}}$$

Tre livelli: propagazione alla risonanza con EIT

Andamento temporale



$$z=100 \text{ m};$$

$$\alpha_0=0.02$$

$$\gamma_{12} = \gamma_{13} / 10^4$$

Massimo dell'impulso

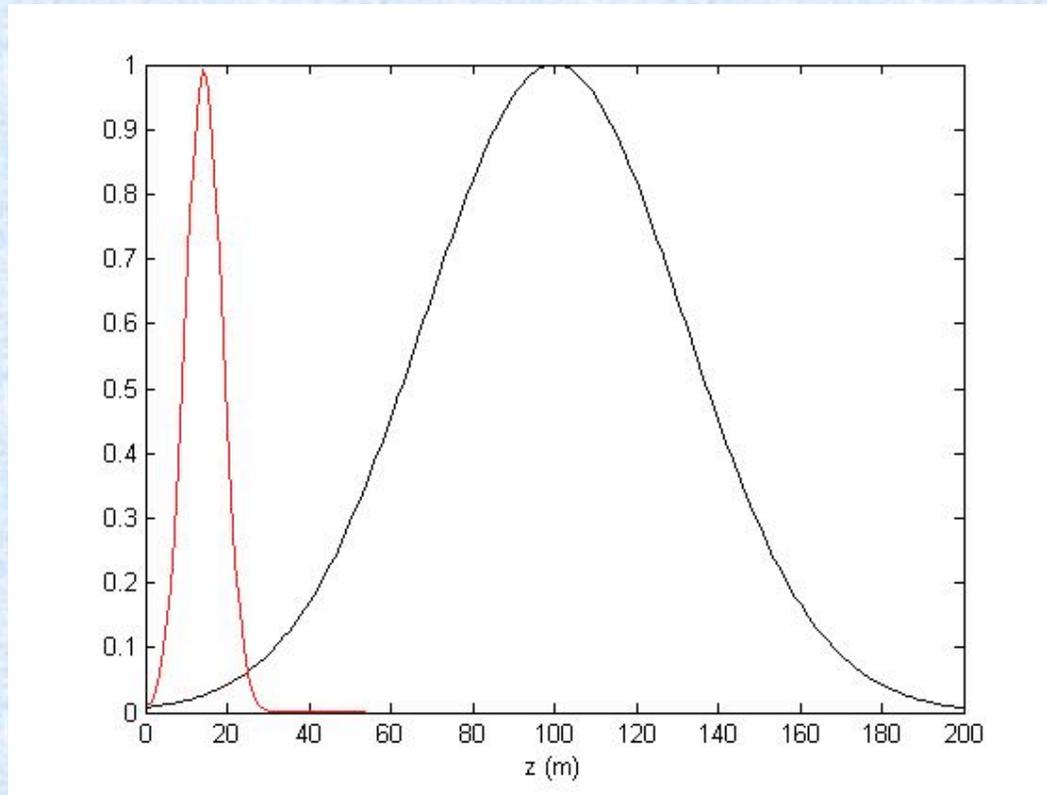
$$I_a(t,z) \quad t_{\max}=333 \text{ ns}$$

$$I_a(t,z) \quad t_{\max}= 2333 \text{ ns}$$

$$u_g/c=1/7$$

Tre livelli: propagazione alla risonanza con EIT

Andamento spaziale



$$t=333 \text{ ns}$$

$$\alpha_0=0.02$$

$$\gamma_{12}=\gamma_{13}/10^4$$

$$\Omega_b=\gamma_{13}/5$$

Massimo dell'impulso

$$I_a(t,z)_{t_{\max}=333 \text{ ns}}$$

$$I_a(t,z)_{t_{\max}=2333 \text{ ns}}$$

Larghezza spaziale

(FWHM)

$$\Delta z=74.6 \text{ m}$$

$$\Delta z=10.6 \text{ m}$$

$$\Delta z/\Delta z=u_g/c$$