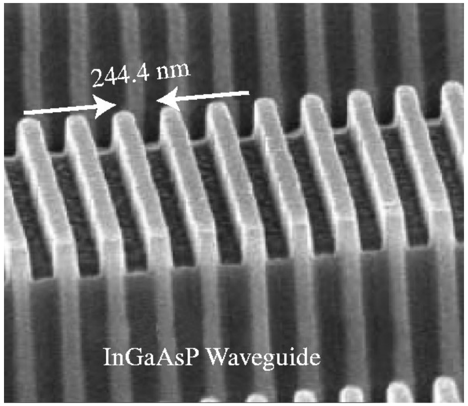
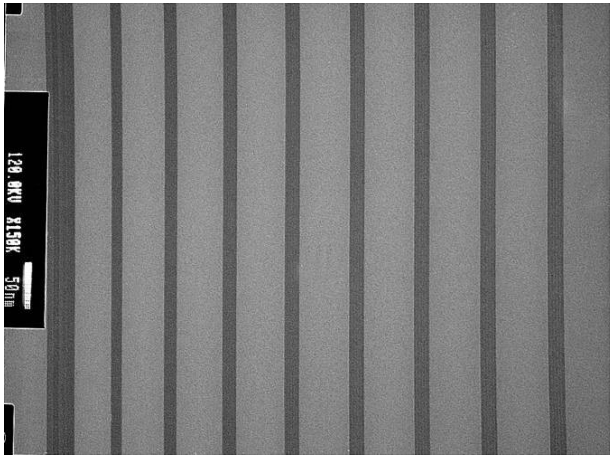
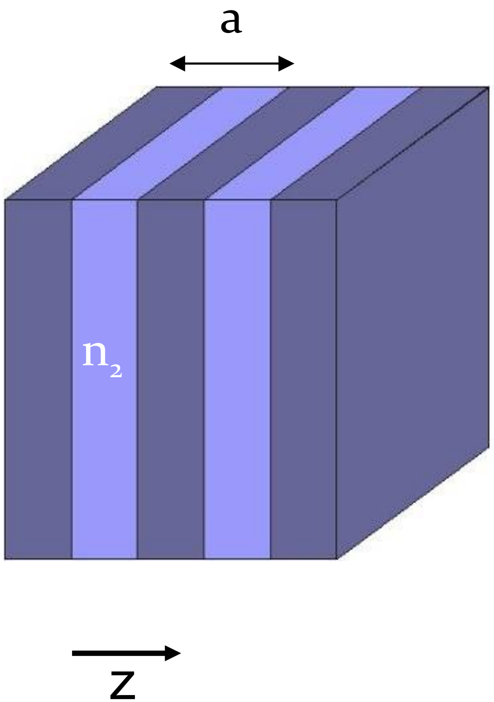
Fotonica 1D
Bragg mirror

PhC in 1D





1D



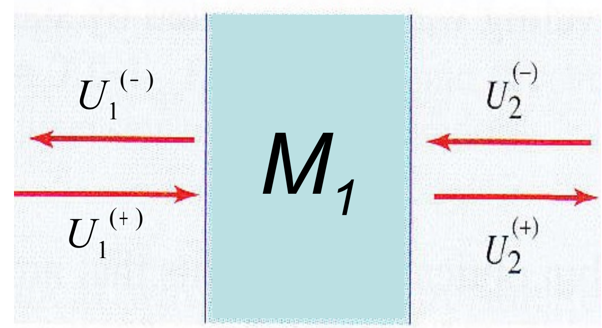
$$\vec{H}_{n,\vec{k}}(\vec{r}) = e^{ik_z z} e^{i\vec{k}_{//}\cdot\vec{\rho}}\,\vec{u}_{n,k_z}(z)$$

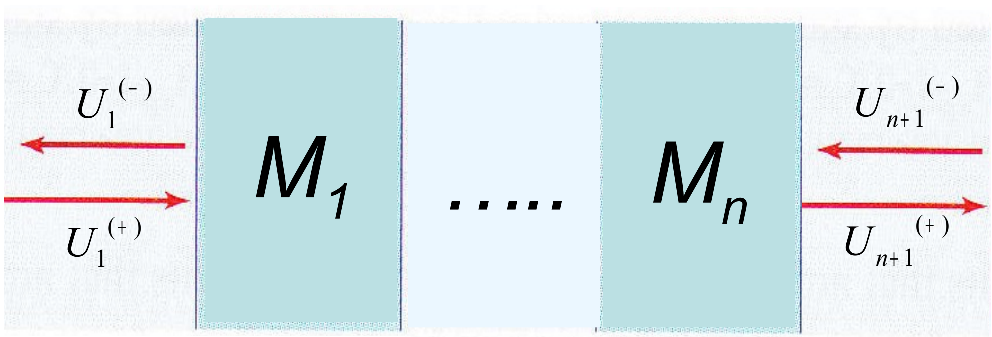
$$\vec{k} = \vec{k}_{//} + \vec{k}_z \qquad \vec{\rho} = x\hat{x} + y\hat{y}$$

$$FBZ \quad -\frac{\pi}{a} \le \left|\vec{k}_z\right| \le \frac{\pi}{a}$$

Metodo matrici M

$$\begin{bmatrix} U_2^{(+)} \\ U_2^{(-)} \end{bmatrix} = M_1 \begin{bmatrix} U_1^{(+)} \\ U_1^{(-)} \end{bmatrix}$$

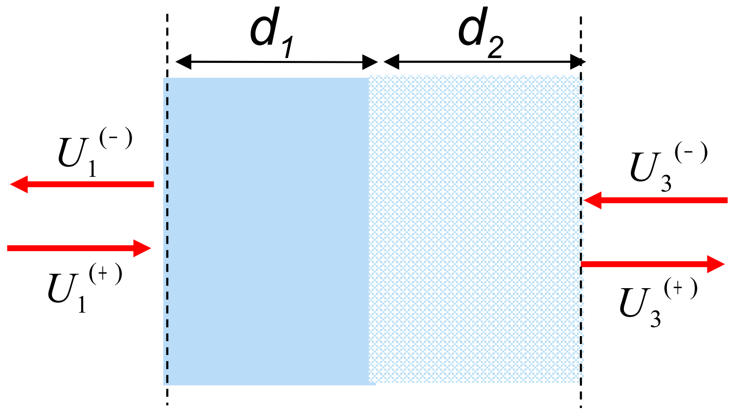




$$\begin{bmatrix} U_2^{(+)} \\ U_2^{(-)} \end{bmatrix} = M_1 \begin{bmatrix} U_1^{(+)} \\ U_1^{(-)} \end{bmatrix} \quad ; \quad \begin{bmatrix} U_3^{(+)} \\ U_3^{(-)} \end{bmatrix} = M_2 \begin{bmatrix} U_2^{(+)} \\ U_2^{(-)} \end{bmatrix} \quad ; \ldots\ldots \quad \begin{bmatrix} U_{n+1}^{(+)} \\ U_{n+1}^{(-)} \end{bmatrix} = M_n \begin{bmatrix} U_n^{(+)} \\ U_n^{(-)} \end{bmatrix}$$

$$\begin{bmatrix} U_{n+1}^{(+)} \\ U_{n+1}^{(-)} \end{bmatrix} = M_n \ldots M_2 M_1 \begin{bmatrix} U_1^{(+)} \\ U_1^{(-)} \end{bmatrix} = M \begin{bmatrix} U_1^{(+)} \\ U_1^{(-)} \end{bmatrix}$$

Propagazione attraverso un mezzo omogeneo seguita da una slab dielettrica



$$M = \begin{bmatrix} \frac{n_2+n_1}{2n_2}\exp(j\varphi_2) & \frac{n_1-n_2}{2n_2}\exp(-j\varphi_2) \\ \frac{n_2-n_2}{2n_2}\exp(j\varphi_2) & \frac{n_2+n_1}{2n_2}\exp(-j\varphi_2) \end{bmatrix} \quad ; \quad \varphi_i = \frac{2\pi}{\lambda} n_i d_i$$

$$\times \begin{bmatrix} \frac{n_2+n_1}{2n_1}\exp(j\varphi_1) & \frac{n_2-n_1}{2n_1}\exp(-j\varphi_1) \\ \frac{n_2-n_1}{2n_1}\exp(j\varphi_1) & \frac{n_2+n_1}{2n_1}\exp(-j\varphi_1) \end{bmatrix} = \begin{bmatrix} \frac{1}{t^*} & \frac{r}{t} \\ \frac{r^*}{t^*} & \frac{1}{t} \end{bmatrix}$$

Propagazione attraverso un mezzo omogeneo seguita da una slab dielettrica

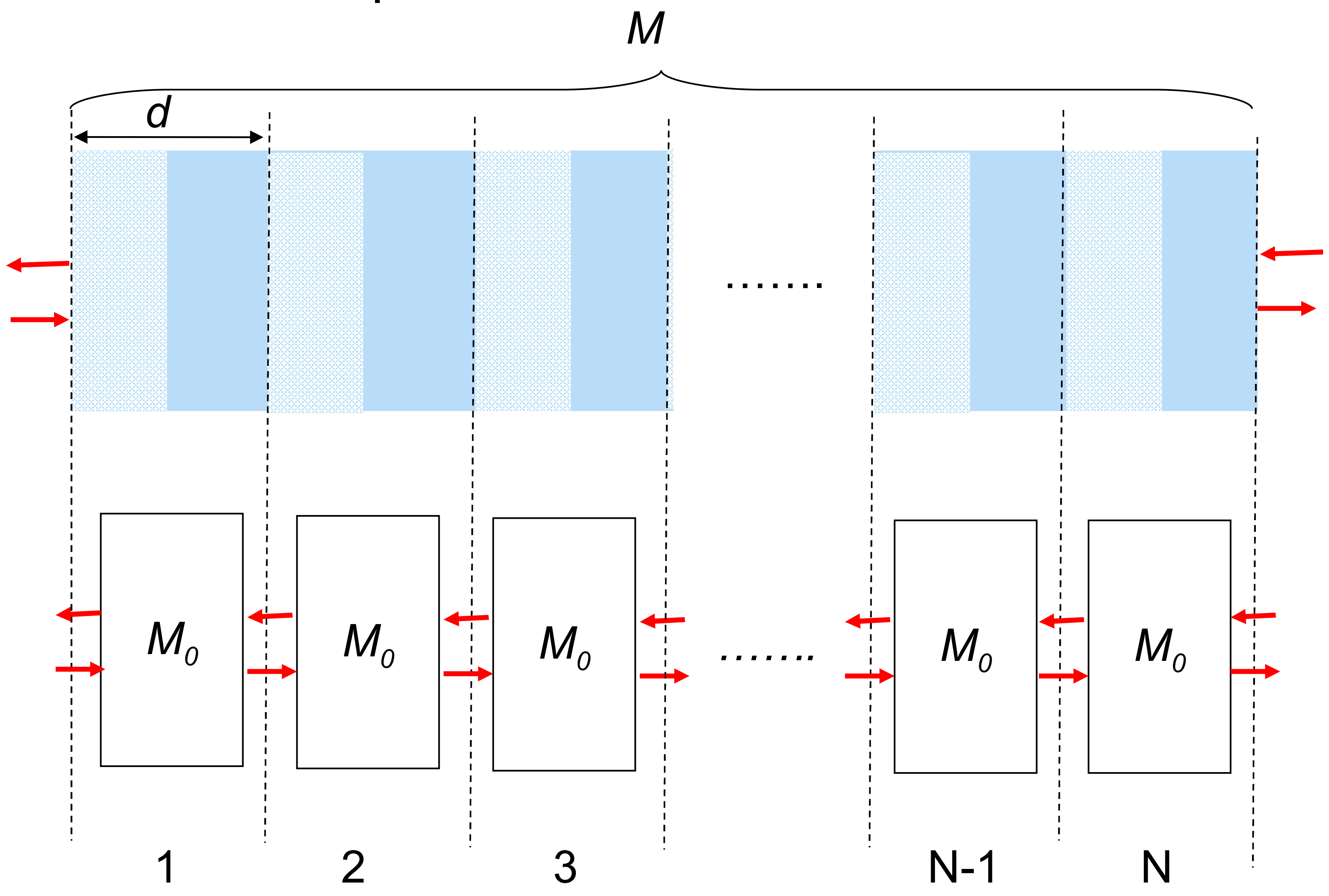
$$\frac{1}{t^*} = \frac{\left(n_2 + n_1\right)^2}{4n_2n_1} \exp j\left(\varphi_2 + \varphi_1\right) - \frac{\left(n_2 - n_1\right)^2}{4n_2n_1} \exp j\left(-\varphi_2 + \varphi_1\right) =$$

$$= \frac{\exp\left(j\varphi_1\right)}{4n_2n_1}\left[\left(n_2 + n_1\right)^2 \exp\left(j\varphi_2\right) - \left(n_2 - n_1\right)^2 \exp\left(-j\varphi_2\right)\right]$$

$$t = \frac{4n_2n_1 \exp\left(j\varphi_1\right)}{\left[\left(n_2 + n_1\right)^2 \exp\left(-j\varphi_2\right) - \left(n_2 - n_1\right)^2 \exp\left(j\varphi_2\right)\right]} =$$

$$= \frac{4n_2n_1 \exp\left(j\varphi_1\right)}{\left[-2j\left(n_2^{\,2} + n_1^{\,2}\right)\sin\left(\varphi_2\right) + 4n_2n_1\cos\left(\varphi_2\right)\right]}$$

Sistema con N periodi



$$M = M_0^{\,N}$$

$N = 1 \qquad M_0^{\;1} = M_0$ Sistema con 1 periodi

$N = 2$ Sistema con 2 periodi

$$M_0^{\;2} = \begin{bmatrix} \dfrac{1}{t^*} & \dfrac{r}{t} \\ \dfrac{r^*}{t^*} & \dfrac{1}{t} \end{bmatrix} \begin{bmatrix} \dfrac{1}{t^*} & \dfrac{r}{t} \\ \dfrac{r^*}{t^*} & \dfrac{1}{t} \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{1}{t^{*2}} + \dfrac{|r|^2}{|t|^2} & \dfrac{r}{|t|^2} + \dfrac{r}{t^2} \\ \dfrac{r^*}{|t|^2} + \dfrac{r^*}{t^{*2}} & \dfrac{1}{t^2} + \dfrac{|r|^2}{|t|^2} \end{bmatrix} = \begin{bmatrix} \dfrac{1}{t^{*2}} + \dfrac{1 - |t|^2}{|t|^2} & \dfrac{r}{|t|^2} + \dfrac{r}{t^2} \\ \dfrac{r^*}{|t|^2} + \dfrac{r^*}{t^{*2}} & \dfrac{1}{t^2} + \dfrac{1 - |t|^2}{|t|^2} \end{bmatrix}$$

Sistema con 2 periodi

$$N = 2$$

$$M_0^{\,2} = \begin{bmatrix} \dfrac{1}{t^{*2}} + \dfrac{1}{|t|^2} - 1 & r\left(\dfrac{1}{|t|^2} + \dfrac{1}{t^2}\right) \\ r^*\left(\dfrac{1}{|t|^2} + \dfrac{1}{t^{*2}}\right) & \dfrac{1}{t^2} + \dfrac{1}{|t|^2} - 1 \end{bmatrix} =$$

$$= \begin{bmatrix} \dfrac{1}{t^*} 2\,\mathrm{Re}\left\{\dfrac{1}{t}\right\} - 1 & \dfrac{r}{t} 2\,\mathrm{Re}\left\{\dfrac{1}{t}\right\} \\ \dfrac{r^*}{t^*} 2\,\mathrm{Re}\left\{\dfrac{1}{t}\right\} & \dfrac{1}{t} 2\,\mathrm{Re}\left\{\dfrac{1}{t}\right\} - 1 \end{bmatrix} = 2\,\mathrm{Re}\left\{\dfrac{1}{t}\right\} M_0 - I$$

Sistema con 2 periodi

$$M_0^{\,2} = 2\,\mathrm{Re}\left\{\frac{1}{t}\right\} M_0 - I$$

$$\textit{posto} \quad \cos\Phi = \mathrm{Re}\left\{\frac{1}{t}\right\} \quad \Rightarrow \quad 2\,\mathrm{Re}\left\{\frac{1}{t}\right\} = \frac{\sin 2\Phi}{\sin\Phi}$$

$$\boxed{M_0^{\,2} = \frac{\sin 2\Phi}{\sin\Phi} M_0 - I}$$

Dimostriamo che

Sistema con N periodi

$$M_0^{\;N} = \frac{\sin N\Phi}{\sin\Phi} M_0 - \frac{\sin(N-1)\Phi}{\sin\Phi} I$$

Per ricorrenza

$$M_0^{\;N+1} = M_0 M_0^{\;N} =$$

$$= \frac{\sin N\Phi}{\sin\Phi} M_0^{\;2} - \frac{\sin(N-1)\Phi}{\sin\Phi} M_0 =$$

$$= \frac{\sin N\Phi}{\sin\Phi}\frac{\sin 2\Phi}{\sin\Phi} M_0 - \frac{\sin(N-1)\Phi}{\sin\Phi} M_0 - \frac{\sin N\Phi}{\sin\Phi} I$$

$$= \frac{\sin N\Phi}{\sin\Phi} 2\cos\Phi\, M_0 - \frac{\sin(N-1)\Phi}{\sin\Phi} M_0 - \frac{\sin N\Phi}{\sin\Phi} I$$

Poiché

$$2\cos\Phi\ \sin N\Phi\ -\ \sin\left(N-1\right)\Phi\ =$$

$$=\ 2\cos\Phi\ \sin N\Phi\ -\ \sin N\Phi\ \cos\Phi\ +\ \sin\Phi\ \cos N\Phi\ =$$

$$=\ \sin\left(N+1\right)\Phi$$

segue

$$M_0^{\,N+1} = \frac{\sin\left(N+1\right)\Phi}{\sin\Phi} M_0 - \frac{\sin N\Phi}{\sin\Phi} I$$

e quindi $\forall\ N$

$$M_0^{\,N} = \frac{\sin N\Phi}{\sin\Phi} M_0 - \frac{\sin\left(N-1\right)\Phi}{\sin\Phi} I$$

Quindi

Sistema con N periodi

$$\frac{1}{t_N} = \frac{\sin N\Phi}{\sin\Phi}\frac{1}{t} - \frac{\sin\left(N-1\right)\Phi}{\sin\Phi}$$

$$\frac{r_N}{t_N} = \frac{\sin N\Phi}{\sin\Phi}\frac{r}{t} \qquad \textit{posto} \qquad \Psi_N = \frac{\sin N\Phi}{\sin\Phi}$$

$$\frac{R_N}{T_N} = \Psi_N^2\frac{R}{T} \quad \Rightarrow \quad \frac{1-T_N}{T_N} = \Psi_N^2\frac{1-T}{T}$$

$$\frac{1}{T_N} = \Psi_N^2\frac{1}{T} - \Psi_N^2 + 1 = \frac{T+\Psi_N^2\left(1-T\right)}{T}$$

$$\boxed{T_N = \frac{T}{T+\Psi_N^2\left(1-T\right)} \quad \Rightarrow \quad R_N = \frac{R\Psi_N^2}{\left(1-R\right)+\Psi_N^2 R}}$$

Quindi

Sistema con N periodi

$$R_N = \frac{R\Psi_N^2}{(1-R)+\Psi_N^2 R}$$

$$\Psi_N = \frac{\sin N\Phi}{\sin\Phi} \qquad \cos\Phi = \mathrm{Re}\left\{\frac{1}{t}\right\}$$

$$\mathrm{Re}\left\{\frac{1}{t}\right\} = \mathrm{Re}\left\{\frac{e^{-i\vartheta}}{|t|}\right\} = \frac{\cos\vartheta}{|t|}$$

$$-\frac{1}{|t|} \le \mathrm{Re}\left\{\frac{1}{t}\right\} \le \frac{1}{|t|} \qquad \frac{1}{|t|} \ge 1$$

$$R_N = \frac{R\Psi_N^2}{(1-R) + \Psi_N^2 R} \qquad \Psi_N = \frac{\sin N\Phi}{\sin\Phi}$$

Sistema con N periodi

a)

$$|\cos\Phi| \le 1 \qquad 0 \le \Psi_N^2 \le N^2$$

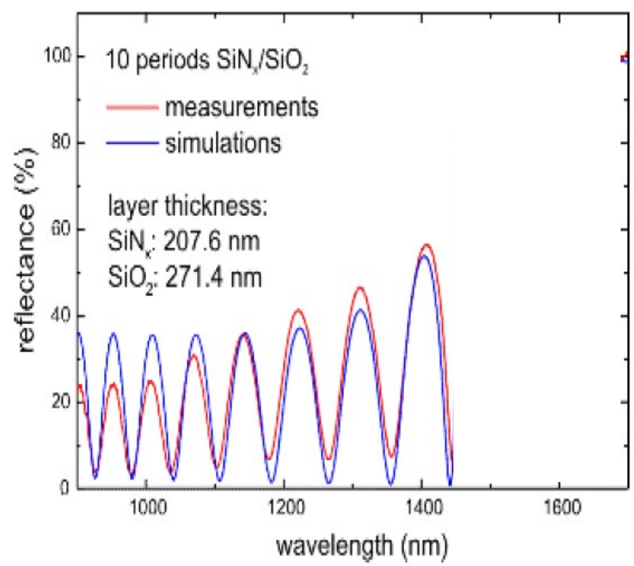
Se

$$\Psi_N^2 = 0 \quad \Rightarrow \quad R_N = 0$$

Se

$$\Psi_N^2 = N^2 \quad \Rightarrow \quad R_N = \frac{N^2 R}{1 - R + N^2 R}$$

Measured and Simulated Reflection Spectra of a Bragg-mirror



$$R_N = \frac{R\Psi_N^2}{(1-R)+\Psi_N^2 R} \qquad \Psi_N = \frac{\sin N\Phi}{\sin\Phi}$$

Sistema con N periodi

b)

$$|\cos\Phi| > 1 \qquad \Phi = \Phi_R + j\Phi_I \qquad \Phi_I \neq 0$$

$$\cos\Phi = \cos\Phi_R \cosh\Phi_I - j\sin\Phi_R \sinh\Phi_I = \mathrm{Re}\left\{\frac{1}{t}\right\}$$

$$\sin\Phi_R = 0 \qquad \Phi_R = m\pi \qquad \cosh\Phi_I = \pm\,\mathrm{Re}\left\{\frac{1}{t}\right\}$$

$$\Psi_N = \frac{\sin N\Phi}{\sin\Phi} = \frac{\cos N\Phi_R}{\cos\Phi_R}\frac{\sinh N\Phi_I}{\sinh\Phi_I} = \pm\frac{\sinh N\Phi_I}{\sinh\Phi_I}$$

b) $$\Psi_N = \pm \frac{\sinh N\Phi_I}{\sinh \Phi_I} \xrightarrow[N\to\infty]{} e^{(N-1)\Phi_I} \to \infty$$

$$R_N = \frac{R\Psi_N^2}{(1-R)+\Psi^2 R} \xrightarrow[N\to\infty]{} 1$$

99.99% riflessione

Leaky modes

Sistema con N periodi

$$R_N = \frac{R\Psi_N^2}{(1-R) + \Psi_N^2 R}$$

$$t = \frac{4 n_2 n_1 \exp(j\varphi_1)}{\left[-2j\left(n_2^2 + n_1^2\right)\sin(\varphi_2) + 4 n_2 n_1 \cos(\varphi_2)\right]}$$

$$\Psi_N = \frac{\sin N\Phi}{\sin\Phi}$$

$$\cos\Phi = \mathrm{Re}\left\{\frac{1}{t}\right\}$$

Addendum for $\left|\mathrm{Re}\left\{\frac{1}{t}\right\}\right| > 1$

$$posto \quad \cos\Phi = \mathrm{Re}\left\{\frac{1}{t}\right\} \quad \Rightarrow \quad \Phi = \Phi_R + i\Phi_I$$

$$\cos\Phi = \cos(\Phi_R + i\Phi_I) = \cos\Phi_R \cosh\Phi_I - i(\sin\Phi_R \sin\Phi_I)$$

$$\sin\Phi_R = 0$$

$$\cosh\Phi_I = \mathrm{Re}\left\{\frac{1}{t}\right\} \qquad if \quad \mathrm{Re}\left\{\frac{1}{t}\right\} > 1$$

$$\cosh\Phi_I = -\mathrm{Re}\left\{\frac{1}{t}\right\} \qquad if \quad \mathrm{Re}\left\{\frac{1}{t}\right\} < -1$$

Dimostriamo la formula per

$$\mathrm{Re}\left\{\frac{1}{t}\right\} > 1 \qquad \Rightarrow \qquad \cosh\Phi_I = \mathrm{Re}\left\{\frac{1}{t}\right\}$$

$$M_0^{\,2} = 2\,\mathrm{Re}\left\{\frac{1}{t}\right\} M_0 - I$$

$$\textit{posto} \quad \cosh\Phi = \mathrm{Re}\left\{\frac{1}{t}\right\} \quad \Rightarrow \quad 2\,\mathrm{Re}\left\{\frac{1}{t}\right\} = \frac{\sinh 2\Phi}{\sinh\Phi}$$

$$M_0^{\,2} = \frac{\sinh 2\Phi}{\sinh\Phi} M_0 - I$$

Essendo

$$\boxed{\sinh(\alpha \pm \beta) = \sinh\alpha\,\cosh\beta \pm \cosh\alpha\,\sinh\beta}$$

Dimostriamo che

Sistema con N periodi

$$M_0^{\,N} = \frac{\sinh N\Phi}{\sinh \Phi} M_0 - \frac{\sinh (N-1)\Phi}{\sinh \Phi} I$$

Per ricorrenza

$$M_0^{\,N+1} = M_0 M_0^{\,N} =$$

$$= \frac{\sinh N\Phi}{\sinh \Phi} M_0^{\,2} - \frac{\sinh (N-1)\Phi}{\sinh \Phi} M_0 =$$

$$= \frac{\sinh N\Phi}{\sinh \Phi} \frac{\sinh 2\Phi}{\sinh \Phi} M_0 - \frac{\sinh (N-1)\Phi}{\sinh \Phi} M_0 - \frac{\sinh N\Phi}{\sinh \Phi} I$$

$$= \frac{\sinh N\Phi}{\sinh \Phi} 2\cos\Phi \, M_0 - \frac{\sinh (N-1)\Phi}{\sinh \Phi} M_0 - \frac{\sinh N\Phi}{\sinh \Phi} I$$

Sistema con N periodi

Poiché

$$2\cosh\Phi\ \sinh N\Phi - \sinh(N-1)\Phi =$$

$$= 2\cosh\Phi\ \sinh N\Phi - \sinh N\Phi\ \cosh\Phi + \sinh\Phi\ \cosh N\Phi =$$

$$= \sinh(N+1)\Phi$$

segue

$$M_0^{N+1} = \frac{\sinh(N+1)\Phi}{\sinh\Phi} M_0 - \frac{\sinh N\Phi}{\sinh\Phi} I$$

e quindi $\forall N$

$$M_0^{N} = \frac{\sinh N\Phi}{\sinh\Phi} M_0 - \frac{\sinh(N-1)\Phi}{\sinh\Phi} I$$

Dimostriamo la formula per

$$\mathrm{Re}\left\{\frac{1}{t}\right\} < -1 \qquad \Rightarrow \qquad \cosh\Phi_I = -\mathrm{Re}\left\{\frac{1}{t}\right\}$$

$$M_0^{\,2} = 2\,\mathrm{Re}\left\{\frac{1}{t}\right\} M_0 - I$$

$$posto \quad \cosh\Phi = -\mathrm{Re}\left\{\frac{1}{t}\right\} \quad \Rightarrow \quad 2\,\mathrm{Re}\left\{\frac{1}{t}\right\} = -\frac{\sinh 2\Phi}{\sinh\Phi}$$

$$M_0^{\,2} = -\frac{\sinh 2\Phi}{\sinh\Phi} M_0 - I$$

Essendo

$$\boxed{\sinh(\alpha \pm \beta) = \sinh\alpha\,\cosh\beta \pm \cosh\alpha\,\sinh\beta}$$

Dimostriamo la formula per

$$M_0^{\,2} = -\frac{\sinh 2\Phi}{\sinh\Phi} M_0 - I = -2\cosh\Phi\, M_0 - I$$

$$M_0^{\,3} = -\frac{\sinh 2\Phi}{\sinh\Phi} M_0^{\,2} - M_0 =$$

$$= -\frac{\sinh 2\Phi}{\sinh\Phi}\left(-2\cosh\Phi\, M_0 - I\right) - M_0 =$$

$$= \left(\frac{2\sinh 2\Phi\;\cosh\Phi}{\sinh\Phi} - \frac{\sinh\Phi}{\sinh\Phi}\right) M_0 + \frac{\sinh 2\Phi}{\sinh\Phi} I =$$

$$= \frac{\sinh 3\Phi}{\sinh\Phi} M_0 + \frac{\sinh 2\Phi}{\sinh\Phi} I$$

$$M_0^{\,3} = \frac{\sinh 3\Phi}{\sinh \Phi} M_0 + \frac{\sinh 2\Phi}{\sinh \Phi} I$$

$$M_0^{\,4} = \frac{\sinh 3\Phi}{\sinh \Phi} M_0^{\,2} + \frac{\sinh 2\Phi}{\sinh \Phi} M_0 =$$

$$= \frac{\sinh 3\Phi}{\sinh \Phi}\left(-\ 2\cosh \Phi\ M_0 -\ I\right) + \frac{\sinh 2\Phi}{\sinh \Phi} M_0 =$$

$$= -\left(\frac{2\sinh 3\Phi\ \ \cosh \Phi}{\sinh \Phi} - \frac{\sinh 2\Phi}{\sinh \Phi}\right) M_0 - \frac{\sinh 3\Phi}{\sinh \Phi} I =$$

$$= -\ \frac{\sinh 4\Phi}{\sinh \Phi} M_0 - \frac{\sinh 2\Phi}{\sinh \Phi} I$$

Dimostriamo che

Sistema con N periodi

$$M_0^{\,N} = (-1)^N \left(-\frac{\sinh N\Phi}{\sinh \Phi} M_0 - \frac{\sinh(N-1)\Phi}{\sinh \Phi} I \right)$$

Per ricorrenza

$$M_0^{\,N+1} = M_0 M_0^{\,N} =$$

$$= (-1)^N \left(-\frac{\sinh N\Phi}{\sinh \Phi} M_0^{\,2} - \frac{\sinh(N-1)\Phi}{\sinh \Phi} M_0 \right) =$$

$$= (-1)^N \left(\frac{\sinh N\Phi}{\sinh \Phi} \frac{\sinh 2\Phi}{\sinh \Phi} M_0 - \frac{\sinh(N-1)\Phi}{\sinh \Phi} M_0 + \frac{\sinh N\Phi}{\sinh \Phi} I \right)$$

$$= (-1)^N \left(\frac{\sinh N\Phi}{\sinh \Phi} 2\cos\Phi \, M_0 - \frac{\sinh(N-1)\Phi}{\sinh \Phi} M_0 + \frac{\sinh N\Phi}{\sinh \Phi} I \right)$$

Poiché

Sistema con N periodi

$$2\cosh\Phi\ \sinh N\Phi - \sinh(N-1)\Phi =$$

$$= 2\cosh\Phi\ \sinh N\Phi - \sinh N\Phi\ \cosh\Phi + \sinh\Phi\ \cosh N\Phi =$$

$$= \sinh(N+1)\Phi$$

segue

$$M_0^{N+1} = (-1)^N\left(\frac{\sinh(N+1)\Phi}{\sinh\Phi}M_0 + \frac{\sinh N\Phi}{\sinh\Phi}I\right)$$

e quindi $\forall N$

$$M_0^N = (-1)^N\left(-\frac{\sinh N\Phi}{\sinh\Phi}M_0 - \frac{\sinh(N-1)\Phi}{\sinh\Phi}I\right)$$

Quindi

Sistema con N periodi

$$M_0^{\,N} = (-1)^N\left(-\frac{\sinh N\Phi}{\sinh\Phi}M_0 - \frac{\sinh\left(N-1\right)\Phi}{\sinh\Phi}I\right)$$

ovvero

$$M_0^{\,N} = (-1)^{N+1}\frac{\sinh N\Phi}{\sinh\Phi}M_0 - (-1)^N\frac{\sinh\left(N-1\right)\Phi}{\sinh\Phi}I$$

posto $\Psi_N = (-1)^{N+1}\dfrac{\sinh N\Phi}{\sinh\Phi}$

$$M_0^{\,N} = \Psi_N M_0 - \Psi_{N-1} I$$