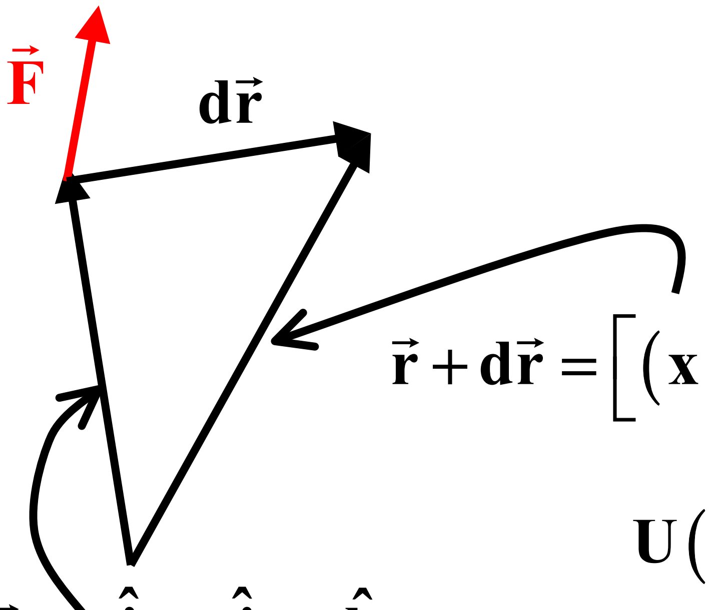
Energia Potenziale, alcune proprietà:

1) Il lavoro elementare

$$\vec{F}(x,y,z)\cdot d\vec{r} = F_x(x,y,z)\,dx + F_y(x,y,z)\,dy + F_z(x,y,z)\,dz$$



$$\vec{r} + d\vec{r} = \left[(x+dx)\hat{i} + (y+dy)\hat{j} + (z+dz)\hat{k}\right]$$

$$U(x_A, y_A, z_A) - U(x_B, y_B, z_B) = L_{A\to B}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\vec{F}(x,y,z)\cdot d\vec{r} = U(x,y,z) - U(x+dx, y+dy, z+dz)$$

Uno spostamento elementare lungo x

$$\vec{F}(x,y,z)\cdot d\vec{r} \equiv F_x(x,y,z)\,dx = U(x,y,z) - U(x+dx,y,z)$$

$$F_x(x,y,z) = \frac{U(x,y,z) - U(x+dx,y,z)}{dx} \rightarrow -\frac{dU(x,y,z)}{dx} \equiv -\frac{\partial U}{\partial x}$$

La derivata "parziale": 1) fissa il valore di y e z. Allora U è funzione solo di x. 2) Fanne la derivata ordinaria

$$F_x = -\frac{\partial U}{\partial x} \quad F_y = -\frac{\partial U}{\partial y} \quad F_z = -\frac{\partial U}{\partial z}$$

$$\vec{F} \equiv -\mathrm{grad}U \equiv -\vec{\nabla}U$$

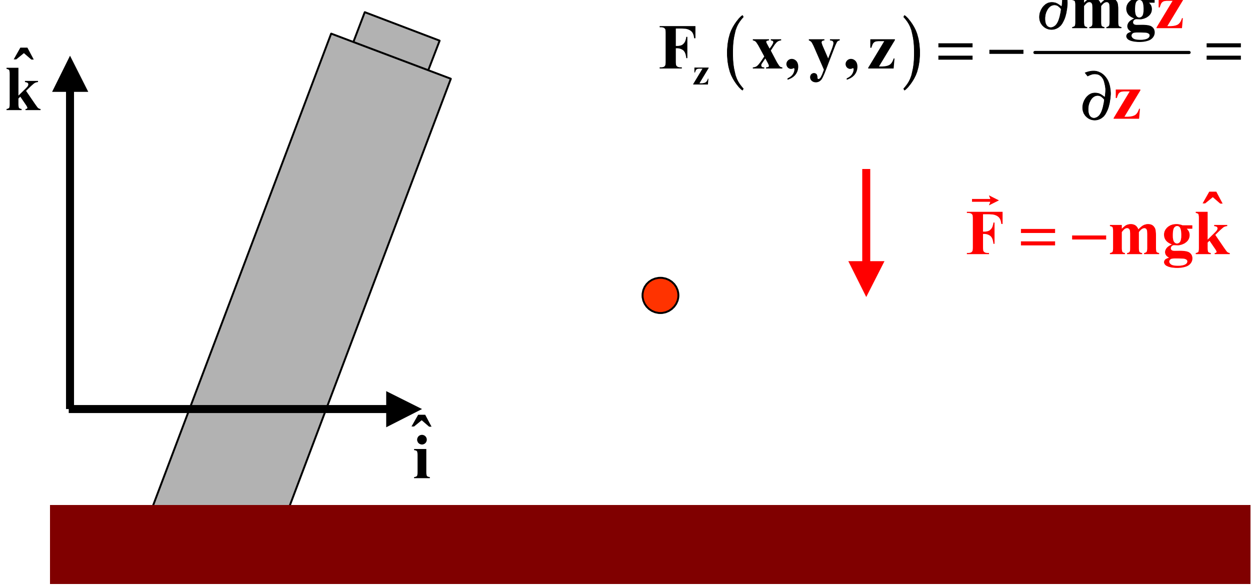
Un po' di esempi:

1) La forza peso

$$U(x,y,z) = mgz$$

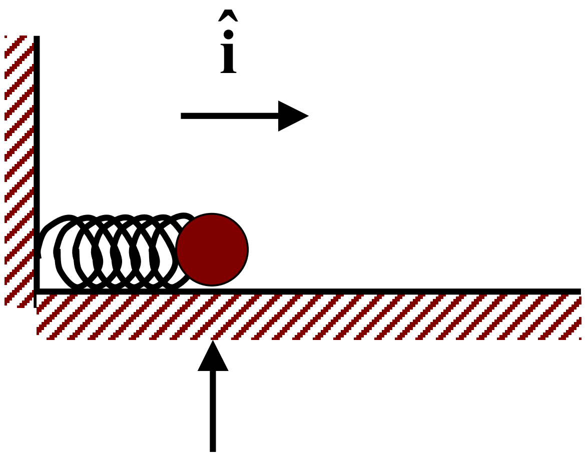
$$F_x(x,y,z) = -\frac{\partial mgz}{\partial x} = 0 \quad F_y(x,y,z) = -\frac{\partial mgz}{\partial y} = 0$$

$$F_z(x,y,z) = -\frac{\partial mgz}{\partial z} = -mg\frac{\partial z}{\partial z} = -mg$$



$$\vec{F} = -mg\hat{k}$$

2) Una molla in una dimensione



x = 0 : molla scarica

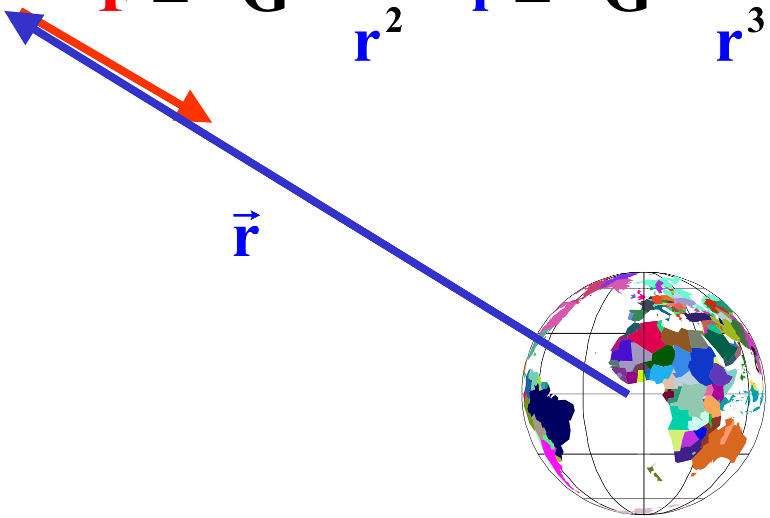
$$U(x,y,z) = U(x) = \frac{1}{2}kx^2$$

$$F_{y,z} = -\frac{\partial U(x)}{\partial y, z} = 0$$

$$F_x = -\frac{\partial \frac{1}{2}kx^2}{\partial x} = -kx$$

3) La gravitazione universale

$$\vec{F} = -G\frac{mM_\oplus}{r^2}\hat{r} = -G\frac{mM_\oplus}{r^3}\vec{r}$$



$$U(x,y,z) = -G\frac{M_\oplus m}{r} = -G\frac{M_\oplus m}{\sqrt{x^2+y^2+z^2}}$$

$$-\frac{\partial U(x,y,z)}{\partial y} = GM_\oplus m\frac{\partial}{\partial y}\frac{1}{\sqrt{x^2+y^2+z^2}}$$

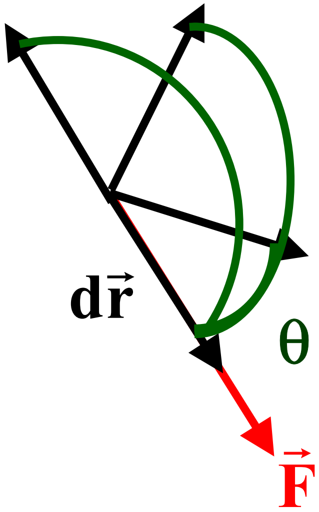
$$= GM_\oplus m\left[-\frac{1}{2}\frac{1}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}2y\right] = -GM_\oplus m\frac{y}{r^3}$$

In totale:

$$F_x = -\frac{GM_{\oplus}m}{r^3}x \quad F_y = -\frac{GM_{\oplus}m}{r^3}y \quad F_z = -\frac{GM_{\oplus}m}{r^3}z$$

$$\vec{F} = -\frac{GM_{\oplus}m}{r^3}\vec{r}$$

Un’ osservazione: il vettore forza indica la direzione di massima diminuzione dell’energia potenziale

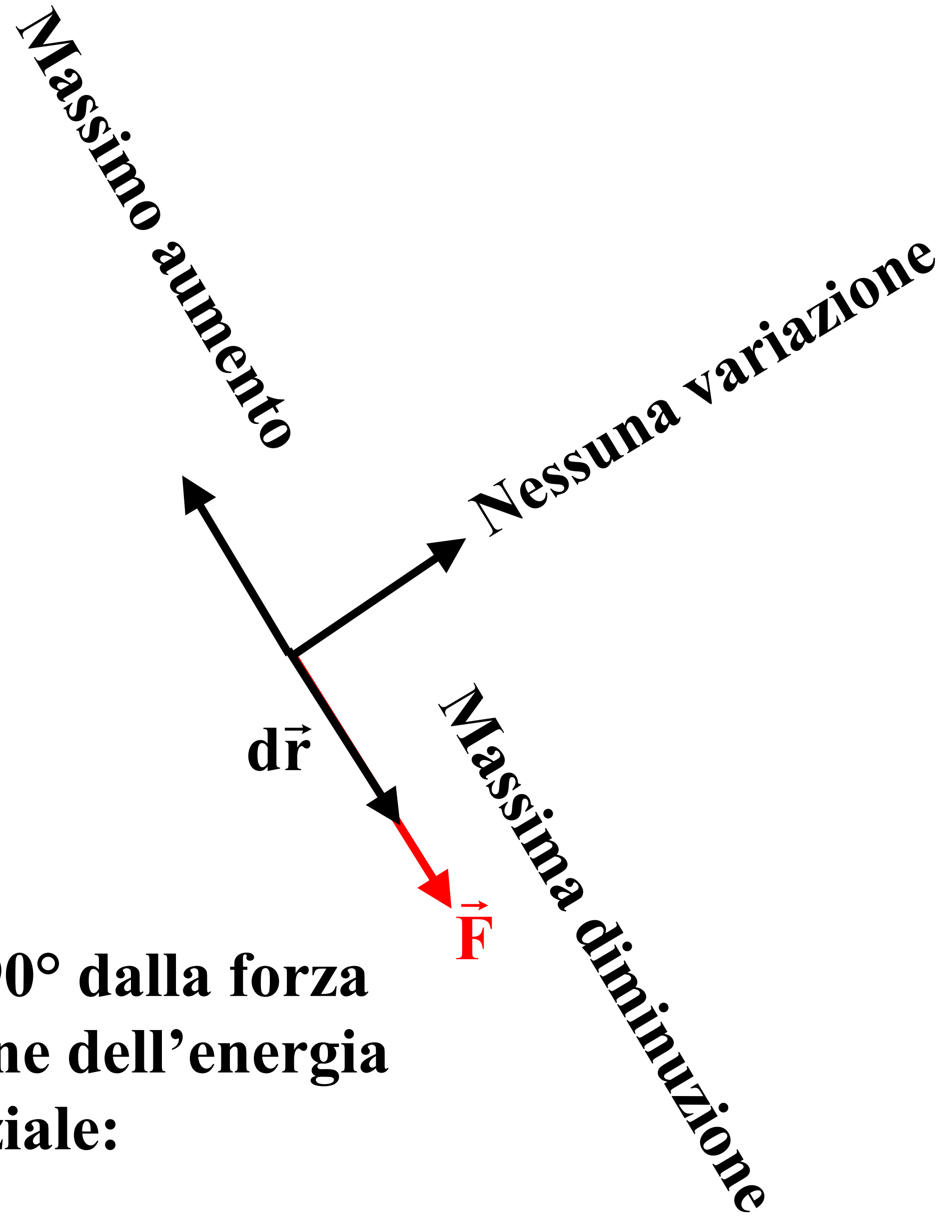


$$\vec{F}(x,y,z)\cdot d\vec{r} = U(x,y,z) - U(x+dx,y+dy,z+dz)$$

$$\vec{F}(x,y,z)\cdot d\vec{r} = \left|\vec{F}(x,y,z)\right|\left|d\vec{r}\right|\mathrm{Cos}\theta$$

Se si varia q a parità di lunghezza dello spostamento

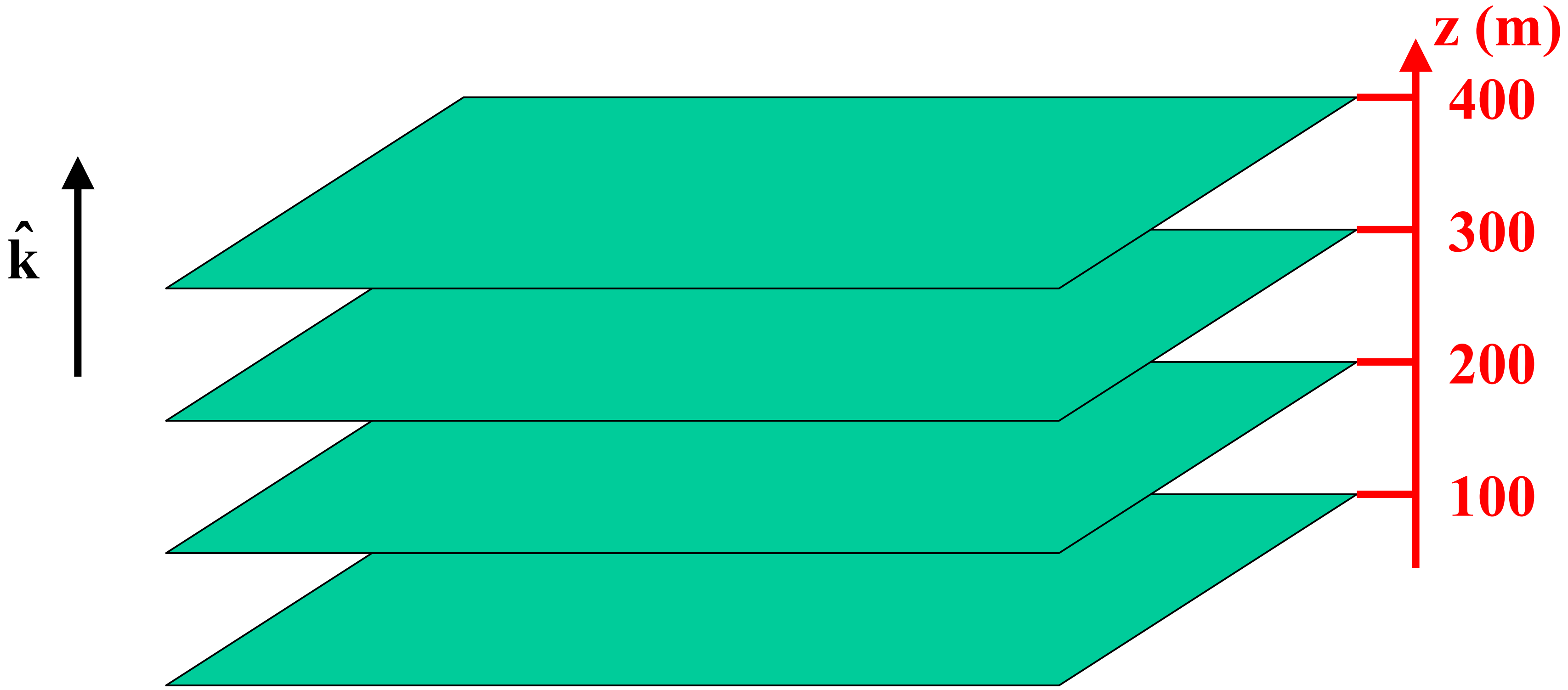
$$\theta = 0 \;\rightarrow \mathrm{Cos}\theta = 1 \;\rightarrow U(x,y,z) - U(x+dx,y+dy,z+dz) \rightarrow \mathrm{Max}$$



Muovendosi a 90° dalla forza non c'è variazione dell'energia potenziale:

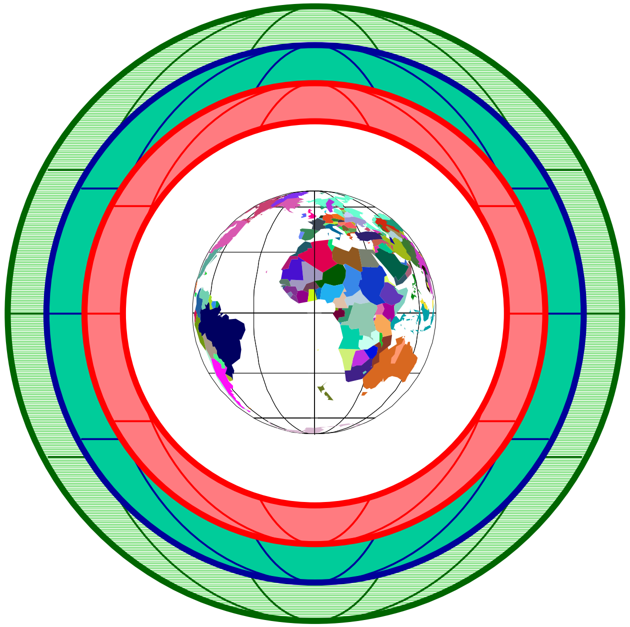
superficie equipotenziale

Superfici equipotenziali $U(x, y, z) = \text{costante}$



La gravità: $mgz = \text{costante} \rightarrow z = \text{costante}$

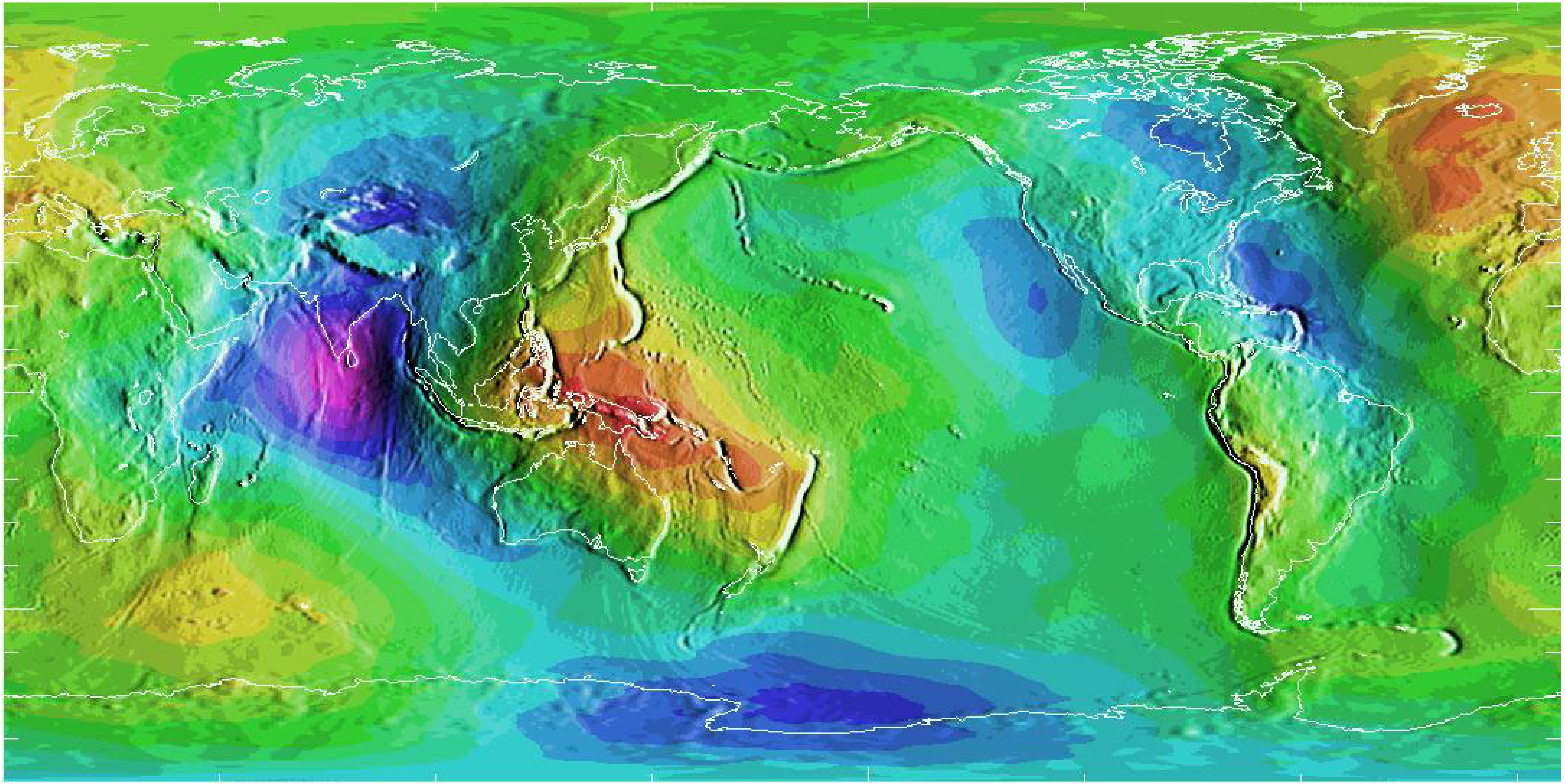
La gravitazione in generale: $U(r) = -\dfrac{GM_{\oplus}m}{r}$



U(r)=costante → r = costante

Le superfici equipotenziali sono sfere concentriche

Color Scale, Upper (Red) : 85.4 meters and higher
Color Scale, Lower (Magenta) :-107.0 meters and lower
Data Max value : 85.4 meters Data Min value :-107.0 meters

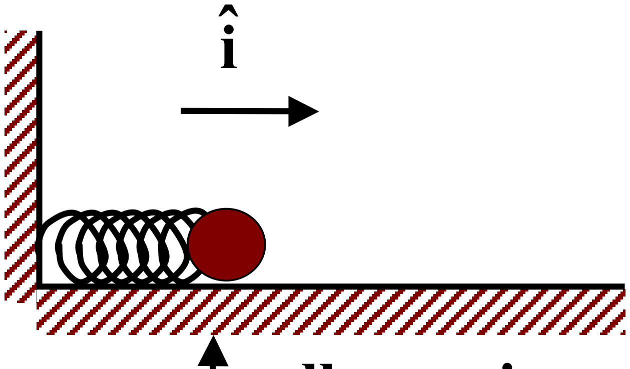


$$-\frac{\mathbf{GM_{\oplus}m}}{\mathbf{r_{\oplus}+h_{vero}+\delta h}}=\mathbf{U(long,lat)}$$

L'energia potenziale in una dimensione $U(x)$

Esempio: la molla

$$U(x) = \frac{1}{2} k (x - x_o)^2$$



$x = x_o$: molla scarica

$$F_x(x) = -\frac{dU}{dx} = -k(x - x_o)$$

$$x = x_o \rightarrow F_x = 0$$

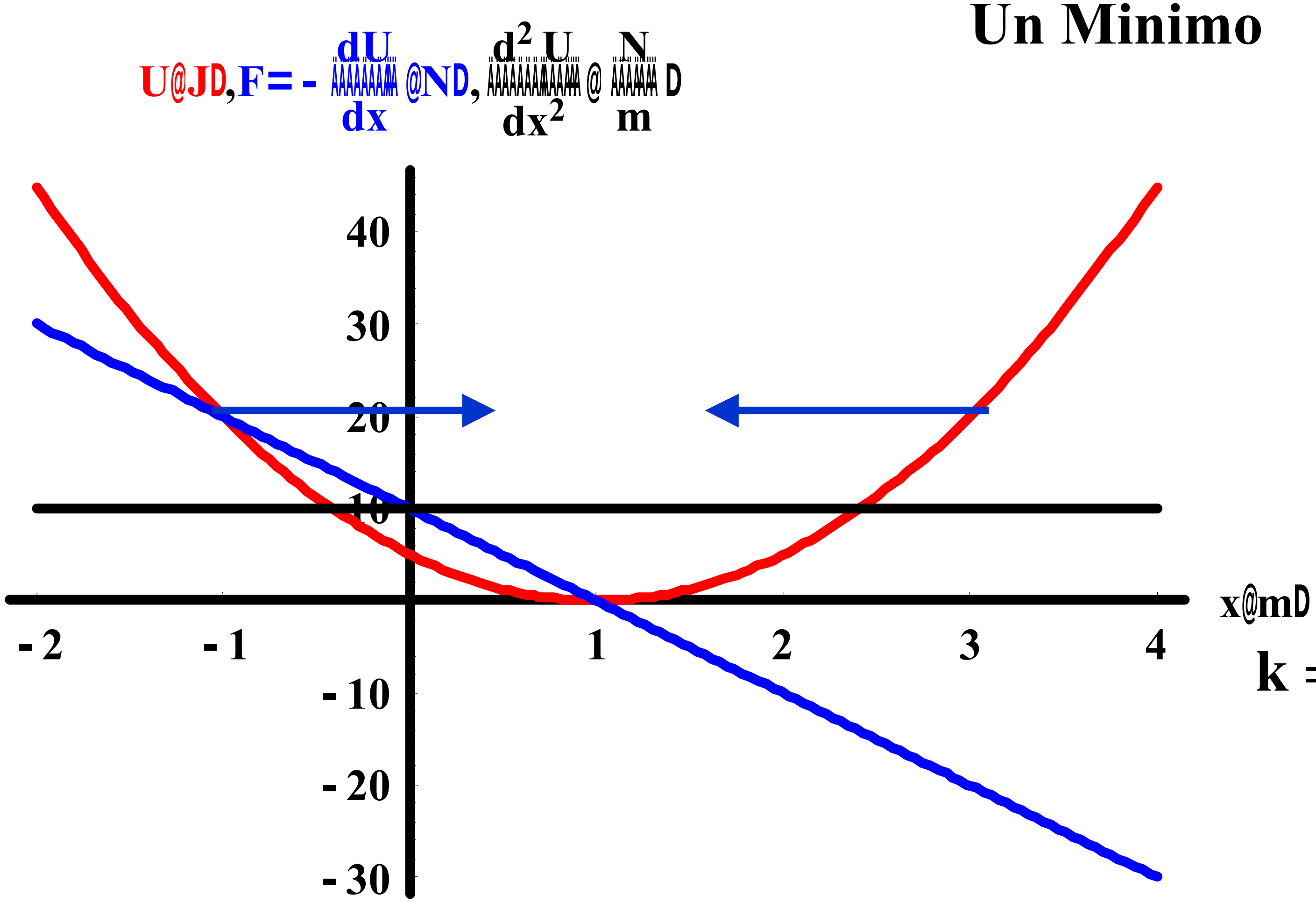
Dall'analisi:

$$\frac{dU}{dx} = 0 \rightarrow \frac{d^2U}{dx^2} > 0 \quad \textbf{Minimo}$$

$$\frac{dU}{dx} = 0 \rightarrow \frac{d^2U}{dx^2} = 0 \quad \textbf{Flesso}$$

$$\frac{dU}{dx} = 0 \rightarrow \frac{d^2U}{dx^2} < 0 \quad \textbf{Massimo}$$

Un Minimo



Molla:

$$m = 1\text{kg}$$

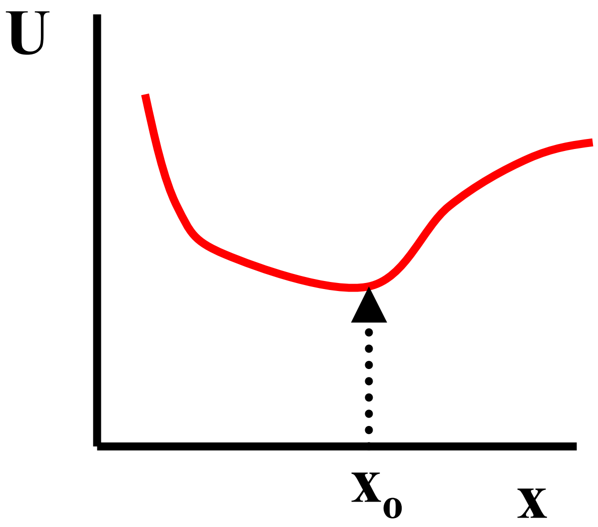
$$k = 10\frac{N}{m}; x_o = +1m$$

$$x = x_o \rightarrow \frac{dU}{dx} = 0, \quad \frac{d^2U}{dx^2} > 0$$

Allontanandosi da x_o nasce una forza che “riporta” in x_o

Equilibrio stabile

Un qualunque minimo è sempre “una molla”:



Una molla di costante elastica

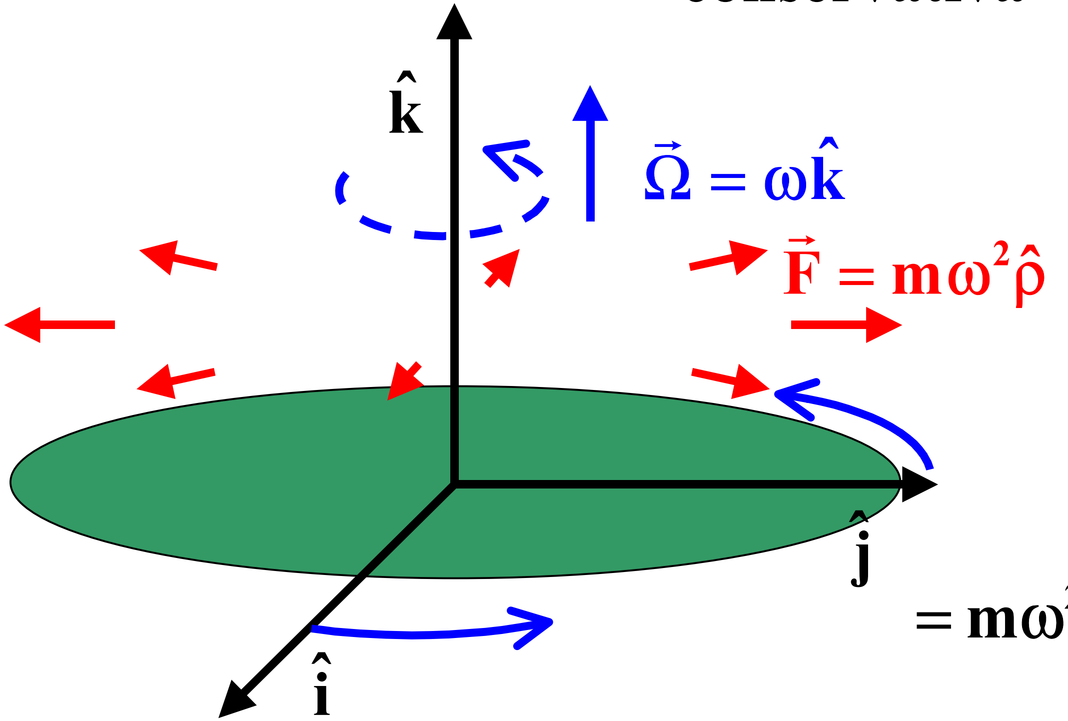
$$k = \left(\frac{d^2U}{dx^2}\right)$$

$$U(x) \approx \cancel{U(x_o)} + \cancel{\left(\frac{dU}{dx}\right)}_{x=x_o}(x - x_o) + \boxed{\frac{1}{2}\left(\frac{d^2U}{dx^2}\right)_{x=x_o}(x - x_o)^2}$$

Si può sempre aggiungere o levare una costante

È 0 in un minimo

Un piccolo esercizio : la forza centrifuga come forza conservativa



$$\vec{F}(x,y,z) = m\omega^2\left(x\hat{i} + y\hat{j}\right)$$

$$L_{A\to B} = \int_{t_A}^{t_B} \vec{F}(t)\cdot\vec{v}(t)\,dt =$$

$$= m\omega^2 \int_{t_A}^{t_B} \left[x(t)\,v_x(t) + y(t)\,v_y(t)\right]dt =$$

$$= m\omega^2 \int_{t_A}^{t_B} \left[x(t)\frac{dx}{dt} + y(t)\frac{dy}{dt}\right]dt =$$

$$= m\omega^2 \int_{t_A}^{t_B} \frac{1}{2}\frac{dx^2 + y^2}{dt}\,dt =$$

$$= \frac{1}{2}m\omega^2\left[x(t_B)^2 + y(t_B)^2\right] - \frac{1}{2}m\omega^2\left[x(t_A)^2 + y(t_A)^2\right]$$

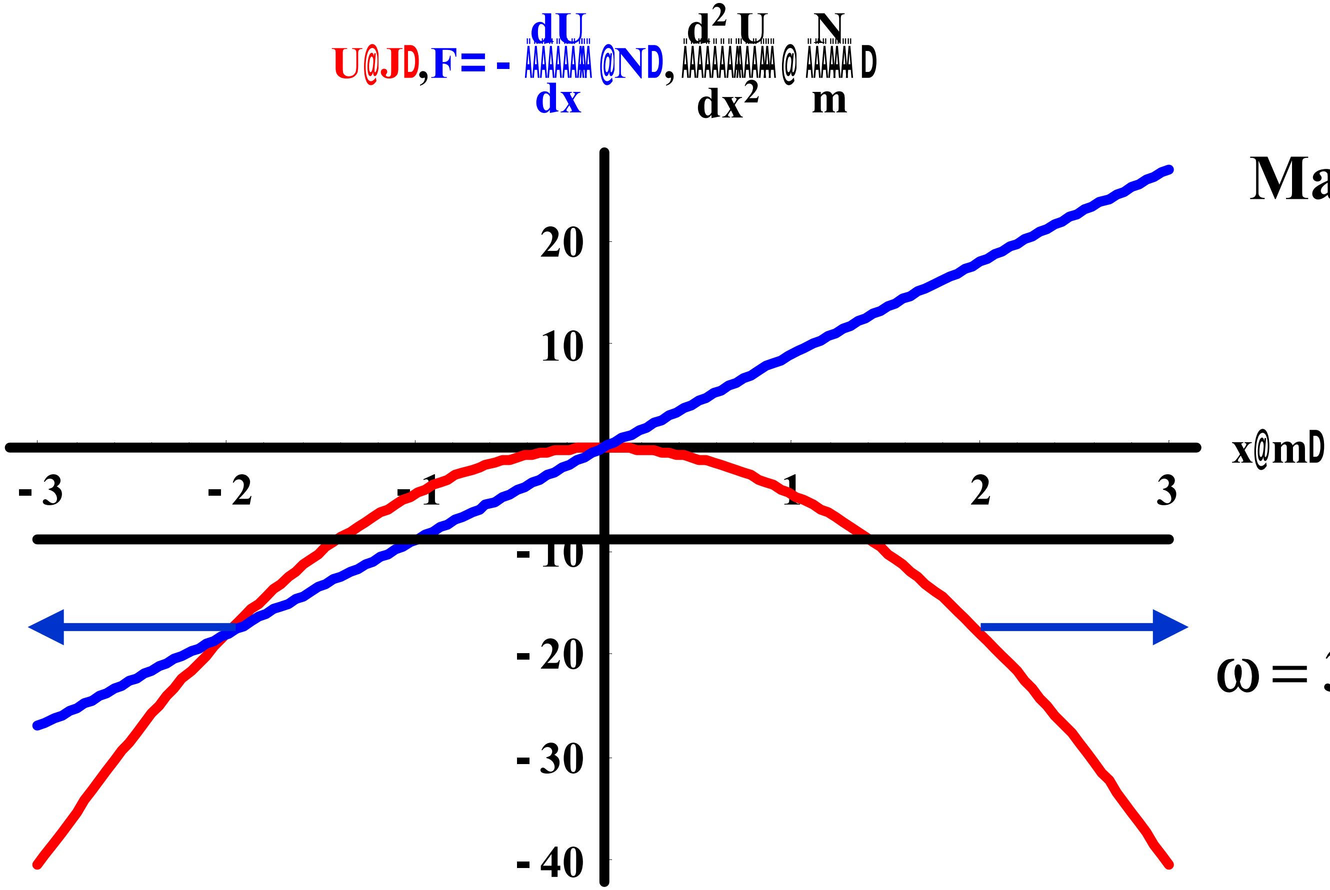
Dunque il lavoro non dipende dal particolare cammino seguito: la forza centrifuga è conservativa!

Energia potenziale:

$$U(x,y,z) = -L_{(x_O,y_O,z_O)\to(x,y,z)} = -\frac{1}{2}m\omega^2\left(x^2+y^2\right)+\frac{1}{2}m\omega^2\left(x_O^2+y_O^2\right)$$

Con $x_O, y_O = 0$

$$U(x,y,z) = -\frac{1}{2}m\omega^2\left(x^2+y^2\right)$$



Un Massimo

$$\omega = 3\frac{\text{rad}}{\text{s}} \qquad m = 1\text{kg}$$

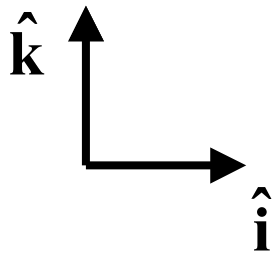
$$x = 0 \rightarrow \frac{dU}{dx} = 0, \quad \frac{d^2U}{dx^2} < 0$$

Allontanandosi da x=0 nasce una forza che "allontana" da x=0

Equilibrio instabile

Un flesso (di ordine infinito):

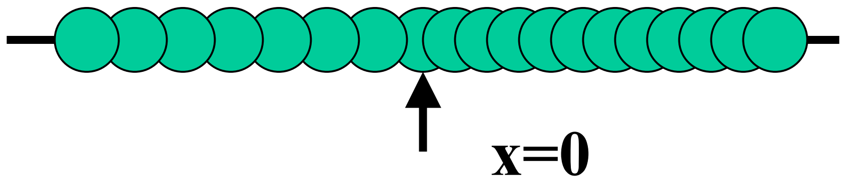
Guida liscia orizzontale



1) Reazione vincolare → nessun lavoro

2) Gravità $U(x,y,z) = mgz$

$$\left(\frac{dU}{dx}\right)_{\text{lungo la guida}} = \frac{\partial U}{\partial x} = 0 \qquad \frac{d^2U}{dx^2} = 0$$

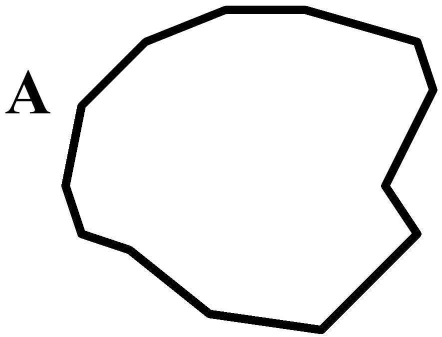


Allontanandosi da x =0 in qualunque direzione non nasce alcuna forza: Equilibrio Indifferente

Ancora alcune osservazioni sull'energia

1) Forze conservative: $$\mathbf{L_{A\to B} = U\left(A\right) - U\left(B\right)}$$

Lavoro su una curva chiusa



$$\mathbf{L_{A\to A} = U\left(A\right) - U\left(A\right) = 0}$$

2) Potenza: lavoro per unità di tempo

(Forze qualunque)

$$dL = \vec{F} \cdot d\vec{r}$$

$$P = \frac{dL}{dt} = \vec{F} \cdot \frac{d\vec{r}}{dt} = \vec{F} \cdot \vec{v}$$

3)Una versione istantanea del teorema dell'energia cinetica (vedi lezioni precedenti)

P_{tot}

$$= \frac{d\,{}^{1}\!/_{2}\,mv^2}{dt}$$

Un esempio

$$z(t) = z(0) + v_z(0)t - \frac{1}{2}gt^2 \qquad x(t) = y(t) = 0$$

$$v_z(t) = v_z(0) - gt \qquad v_x(t) = v_y(t) = 0$$

$$\frac{1}{2}mv^2 = \frac{1}{2}m\left[v(0) - gt\right]^2$$

$$\frac{d(1/2)mv^2}{dt} = \frac{1}{2}m\frac{d}{dt}\left[v(0) - gt\right]^2$$

$$= m\left[v(0) - gt\right]\frac{d}{dt}\left[v(0) - gt\right]$$

$$= m\left[v(0) - gt\right](-g) \quad = -mg\left[v(0) - gt\right]$$

$$\vec{F} = -mg\hat{k} \qquad \vec{F} \cdot \vec{v} = -mg\left[v(0) - gt\right]$$

Teorema dell'energia se sono presenti forze conservative e non

$$\vec{F}_{tot} = \vec{F}_{conservative} + \vec{F}_{non\ conservative}$$

$$L_{tot,A\to B} = \int_A^B \left(\vec{F}_{conservative} + \vec{F}_{non\ conservative}\right)\cdot d\vec{r}$$

$$= \int_A^B \vec{F}_{conservative}\cdot d\vec{r} + \int_A^B \vec{F}_{non\ conservative}\cdot d\vec{r}$$

$$= U(A) - U(B) + \int_A^B \vec{F}_{non\ conservative}\cdot d\vec{r}$$

$$L_{tot,A\to B} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

$$L_{tot,A\rightarrow B} = U(A) - U(B) + \int_A^B \vec{F}_{non\ conservative} \cdot d\vec{r}$$

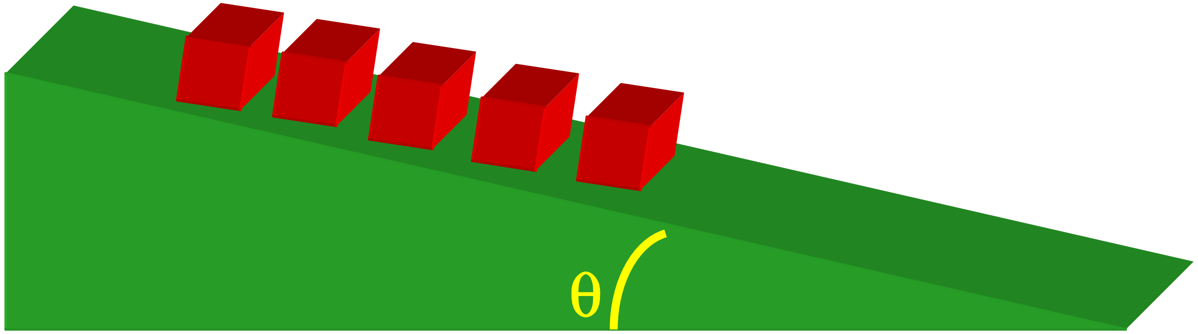
$$L_{tot,A\rightarrow B} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2$$

$$\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = U(A) - U(B) + \int_A^B \vec{F}_{non\ conservative} \cdot d\vec{r}$$

$$\int_A^B \vec{F}_{non\ conservative} \cdot d\vec{r} = \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 - \left[U(A) - U(B)\right]$$

$$= \left[\frac{1}{2}mv_B^2 + U(B)\right] - \left[\frac{1}{2}mv_A^2 + U(A)\right] = E_B - E_A$$

Esempio: piano inclinato con attrito

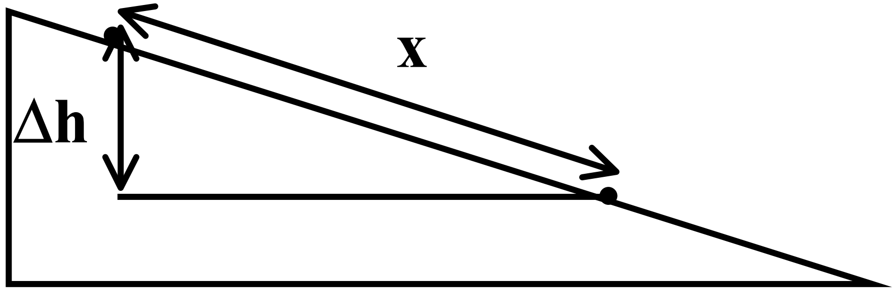


$$m\frac{d^2x}{dt^2} = mg\sin\theta - \mu_d mg\cos\theta$$

Partenza da fermo, x(0) = 0

$$x(t) = \frac{1}{2}g\left(\sin\theta - \mu_d\cos\theta\right)t^2$$

$$v_x(t) = g\left(\sin\theta - \mu_d\cos\theta\right)t$$



$$U(x) = U(0) - mg\Delta h = U(0) - mgx\sin\theta$$

$$E(t) = \frac{1}{2}m\underbrace{\left[g^2\left(\sin\theta-\mu_d\cos\theta\right)^2 t^2\right]}_{v^2}$$

$$+U(0)-mg\sin\theta\underbrace{\left[\frac{1}{2}g\left(\sin\theta-\mu_d\cos\theta\right)t^2\right]}_{\Delta h}$$

$$=\frac{1}{2}m\left[g^2\left(\cancel{\sin^2\theta}-\cancel{2}\mu_d\cos\theta\sin\theta+\mu_d^2\cos^2\theta\right)t^2\right]$$

$$+U(0)-\frac{1}{2}mg^2\left(\cancel{\sin^2\theta}-\cancel{\mu_d\cos\theta\sin\theta}\right)t^2$$

$$=\frac{1}{2}m\left[g^2\left(-\mu_d\cos\theta\sin\theta+\mu_d^2\cos^2\theta\right)t^2\right]+U(0)$$

$$=\underbrace{-\mu_d mg\cos\theta}_{F_{attr}}\underbrace{\frac{1}{2}g\left(\sin\theta-\mu_d\cos\theta\right)t^2}_{x}+U(0)$$

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$$\mathbf{E(t) = F_{attr}\,x\left(t\right) + U\left(0\right)}$$

$$\mathbf{E(t_B) - E(t_A) = F_{attr}\,x\left(t_B\right) + U\left(0\right) - F_{attr}\,x\left(t_A\right) - U\left(0\right)}$$

$$\mathbf{= \underbrace{F_{attr}\left[x\left(t_B\right) - x\left(t_A\right)\right]}_{L_{attr}}}$$