ANALISI 1 - C.d.L. in FISICA A.A.2005/06

Successioni numeriche

Teorema 1 (Criterio del confronto)**.** *Siano* (a_n) *e* (b_n) *due successioni tali che* $a_n \leq b_n$ *definitivamente. Allora*

$$\lim_{n\to+\infty} a_n = +\infty \qquad \Rightarrow \qquad \lim_{n\to+\infty} b_n = +\infty$$
$$\lim_{n\to+\infty} b_n = -\infty \qquad \Rightarrow \qquad \lim_{n\to+\infty} a_n = -\infty.$$

Esempio 2.

$$\left[\frac{n}{2}\right] \to +\infty; \qquad \frac{n!}{2^n} \to +\infty; \qquad \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n+1}} + ... + \frac{1}{\sqrt{2n}} \to +\infty$$

Teorema 3 (Teorema dei due carabinieri)**.** *Siano* (a_n), (b_n) *e* (c_n) *tre successioni tali che*

$$a_n \leq b_n \leq c_n \qquad \forall\, n \in \mathbb{N}.$$

Se esistono e sono uguali i limiti $\lim_{n+\infty} a_n$ *e* $\lim_{n\to+\infty} c_n$*. allora esiste* $\lim_{n+\infty} b_n$ *e*

$$\lim_{n+\infty} b_n = \lim_{n+\infty} a_n = \lim_{n\to+\infty} c_n.$$

Esempio 4.

$$\frac{1}{n^2} + \frac{1}{(n+1)^2} + ... + \frac{1}{(n+n)^2} \to 0; \qquad \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + ... + \frac{1}{\sqrt{n^2+n}} \to 1;$$
$$\frac{1}{\sqrt{n^2}} + \frac{1}{\sqrt{n^2+1}} + ... + \frac{1}{\sqrt{n^2+2n}} \to 2;$$

Teorema 5 (Criterio del rapporto)**.** *Sia* (a_n) *una successione di numeri reali positivi. Allora*

$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \Lambda \in (1, +\infty] \qquad \Rightarrow \qquad \lim_{n\to+\infty} a_n = +\infty$$
$$\lim_{n\to+\infty} \frac{a_{n+1}}{a_n} = \Lambda \in [0, 1) \qquad \Rightarrow \qquad \lim_{n\to+\infty} a_n = 0$$

Esempio 6.

$$\frac{\alpha^n}{n^\beta} \to +\infty, \quad (\alpha > 1, \beta > 0); \qquad \alpha^n n^\beta \to 0, \quad (0 < \alpha < 1, \beta > 0);$$
$$\frac{\alpha^n}{n!} \to 0, \quad (\alpha > 0); \qquad \frac{n^\beta}{n!} \to 0, \quad (\beta > 0);$$
$$\frac{n^n}{n!} \to +\infty.$$

Teorema 7 (Criterio di Cesaro $\lambda/+\infty$)**.** *Siano* (a_n) *e* (b_n) *successioni reali,* $0 < b_n < b_{n+1} \to +\infty$*. Allora*

$$\lim_{n\to+\infty} \frac{a_{n+1} - a_n}{b_{n+1} - b_n} = \Lambda \in \mathbb{R} \cup \{\pm\infty\} \quad \Rightarrow \quad \lim_{n\to+\infty} \frac{a_n}{b_n} = \Lambda.$$

Esempio 8.

$$\frac{\log n}{n} \to 0; \qquad \frac{\log n!}{n} \to +\infty; \qquad \frac{\log n!}{n^2} \to 0;$$

Corollario 9 (Teorema sulla media aritmetica)**.** *Data una successione* a_n*. Allora*

$$\lim_{n\to+\infty} a_n = \lambda \in \mathbb{R} \cup \{\pm\infty\} \quad \Rightarrow \quad \lim_{n\to+\infty} \frac{a_1 + ... + a_n}{n} = \lambda.$$

Esempio 10.

$$\frac{1+\frac{1}{2}+...+\frac{1}{n}}{n} \to 0.$$

Teorema 11 (Teorema sulla media geometrica)**.** *Sia* (a_n) *una successione a termini positivi. Allora*

$$\lim_{n\to+\infty} a_n = \lambda \in \mathbb{R}\cup\{+\infty\} \quad \Rightarrow \quad \lim_{n\to+\infty} \sqrt[n]{a_1\cdot ... \cdot a_n} = \lambda.$$

Teorema 12. *Sia* (a_n) *a termini positivi. Allora*

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lambda \in \mathbb{R}\cup\{+\infty\} \quad \Rightarrow \quad \lim_{n\to+\infty} \sqrt[n]{a_n} = \lambda.$$

Esempio 13.

$$\sqrt[n]{n!} \to +\infty; \qquad \frac{n}{\sqrt[n]{n!}} \to e; \qquad \sqrt[2n]{n!} \to +\infty;$$

$$\frac{n^\alpha}{\sqrt[n]{n!}} \to 0, \quad (0<\alpha<1); \qquad \frac{n^\alpha}{\sqrt[n]{n!}} \to e, \quad (\alpha=1); \qquad \frac{n^\alpha}{\sqrt[n]{n!}} \to +\infty, \quad (\alpha>1);$$

$$\sqrt[n]{\binom{2n}{n}} \to 4$$

Teorema 14 (Criterio di Cesaro 0/0)**.** *Siano* (a_n) *e* (b_n) *successioni reali infinitesime,* (b_n) *strettamente monotona. Allora*

$$\lim_{n\to+\infty} \frac{a_{n+1}-a_n}{b_{n+1}-b_n} = \Lambda \in \mathbb{R}\cup\{\pm\infty\} \quad \Rightarrow \quad \lim_{n\to+\infty} \frac{a_n}{b_n} = \Lambda.$$

Teorema 15 (Criterio della radice)**.** *Sia* (a_n) *una successione di numeri reali positivi. Allora*

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \Lambda \in (1,+\infty] \quad \Rightarrow \quad \lim_{n\to+\infty} a_n = +\infty$$

$$\lim_{n\to+\infty} \sqrt[n]{a_n} = \Lambda \in [0,1) \quad \Rightarrow \quad \lim_{n\to+\infty} a_n = 0$$

Teorema 16 (Stirling)**.**

$$n! = \frac{n^n}{e^n}(\sqrt{2\pi n}+a_n), \quad (con\,|a_n|\le 1).$$

Esercizio 17. Provare che

$$\frac{n^n}{\alpha^n n!} \to 0, \quad (\alpha>e); \qquad \frac{n^n}{\alpha^n n!} \to +\infty, \quad (0<\alpha<e); \qquad \frac{n^n}{(2n)!} \to 0;$$

$$\frac{n!n^n}{(pn)!} \to 0, \quad (p\in\mathbb{N}, p\ge 2); \qquad \frac{(pn)^n n!}{(2n)!} \to +\infty, \quad (p\in\mathbb{N}, p\ge 2); \qquad \frac{(n+p)^n}{(n+q)!} \to +\infty, \quad (p,q\in\mathbb{N}).$$

$$\sqrt[n]{a^n+b^n} \to \max\{a,b\} \quad (a,b\ge 0)$$

Esercizio 18. Calcolare, se esite, il limite delle seguenti successioni, ricordando che

$$\lim_{n\to+\infty} \frac{\log n}{n^b} = 0 \quad \forall b>0, \qquad \lim_{n\to+\infty} \frac{n^b}{a^n} = 0 \quad \forall b>0, \forall a>1, \qquad \lim_{n\to+\infty} \frac{a^n}{n!} = 0 \quad \forall a>1, \qquad \lim_{n\to+\infty} \frac{n!}{n^n} = 0,$$

e che $\lim_{n\to+\infty} \frac{n}{\sqrt[n]{n!}} = e$.

$3n^2 - 2n - 4 \quad (+\infty)$, $\quad -2n^3 + n - 2 \quad (-\infty)$, $\quad -n^5 + 4n^3 - 3 \quad (-\infty)$,

$4n^4 + 5n^3 - 2n^2 - 10n + 1 \quad (+\infty)$, $\quad \frac{n + \cos n}{n + \sin n^2} \quad (1)$, $\quad \frac{n^2 - 2n + 3}{2n + 1} \quad (+\infty)$,

$\frac{n^3 - n - 1}{3 - n} \quad (-\infty)$, $\quad \frac{3n^2 + 1}{-2n^2 + 1} \quad (-\frac{3}{2})$, $\quad \frac{n^3 - 1}{(n+1)(n^2 + 2n + 1)} \quad (1)$,

$\frac{(2n+1)(3n^2 - n^3 + 3)}{(n^2 + 2n + 2)\left(\frac{n}{2} + 1\right)} \quad (-\infty)$, $\quad \frac{\sqrt{n^2 + 2}}{2n + 3} \quad (\frac{1}{2})$, $\quad \frac{\sqrt{n+1}}{\sqrt{n^2 + n + 1} - \sqrt{n}} \quad (0)$,

$\frac{\sqrt{n^2 + 1} - \sqrt{n^3 + 3}}{2\sqrt{n^3 + 1}} \quad (-\frac{1}{2})$, $\quad (-1)^n \frac{n}{n+1} \quad (\nexists)$, $\quad \frac{\sqrt{2n+1} - \sqrt{2n}}{\sqrt{2n+1} + \sqrt{n}} \quad (0)$,

$\frac{n}{1 + (-1)^{n-1} n} \quad (\nexists)$, $\quad \frac{n^{(-1)^n}}{n^{3/2}} \quad (0)$, $\quad \left(1 + \frac{1}{n+2}\right)^{n+3} \quad (e)$,

$\left(1 + \frac{(-1)^n}{n}\right)^n \quad (\nexists)$, $\quad \left(1 + \frac{1}{n^2}\right)^n \quad (1)$, $\quad \left(\frac{n^3 + 1}{n^4 + 3}\right)^n \quad (0)$,

$\left(\frac{1 - n}{n}\right)^n \quad (\nexists)$, $\quad \sqrt[n]{\frac{2^{n+1} + n^2}{5^n - 3^{n+1}}} \quad (\frac{2}{5})$, $\quad \frac{n! - 2n^n}{n^n - n^3} \quad (-2)$,

$\frac{2^{n+2}}{n! - 3^n} \quad (0)$, $\quad \frac{-n^{3/2} + n^{7/8} - n + 2}{\log_2 2^n + n^{7/5}} \quad (-\infty)$, $\quad \left(\frac{n^2 + n - 2}{n^2 + 1}\right)^{\sqrt{n}} \quad (1)$,

$\left(1 - \frac{3\sqrt{n}}{n+2}\right)^{\sqrt[4]{n}} \quad (1)$, $\quad \frac{2^{2n} + 5^{(n+1)/2}}{5^{(n-1)/2} + 2^{n+1}} \quad (+\infty)$, $\quad \left(\frac{\sqrt{n+1}}{\sqrt{n+1} - 1}\right)^{-n} \quad (0)$,

$\frac{2^{-2n} + 2n!}{2^n + n^2} \quad (+\infty)$, $\quad \frac{n}{\log_2(2^n + e)} \quad (1)$, $\quad \sqrt{\log_2 n - \log_3(n-2) + 2} \quad (+\infty)$,

$\left(1 + \frac{1}{\log(e^{n+1} + 3)}\right)^n \quad (e)$, $\quad \left(\frac{n^3 + 1}{n^4 + 3}\right)^{n/2} \quad (0)$, $\quad \left(\frac{1 + \log(n^3)}{3 \log n}\right)^{\log(n^3 + 1)} \quad (e)$,

$\left|(-1)^n \frac{n}{n+1}\right| \quad (1)$, $\quad \frac{\frac{-n^3}{(2+n)^2} + \frac{\arctan((n-1)!)}{4}}{n} \quad (-1)$, $\quad \frac{(2n+5)^{2/3} + (2n-1)^{2/3}}{\sqrt{n^2 + 1} + n^{1/4}} \quad (0)$,

$\frac{2^n + (-1)^n (n+1)^2}{-5^n + n} \quad (0)$, $\quad \sqrt[n]{\pi^n + (-1)^n 3^n} \quad (\pi)$, $\quad \frac{n\sqrt[n]{5^n - 3^n}}{\sqrt[n]{n! - 5^n}} \quad (5e)$.

Esercizio 19. Calcolare, se esiste, il limite delle seguenti successioni.

$$\sqrt[3]{\frac{|\sin n|}{n+1}}; \qquad \frac{\sqrt{n+1}-\sqrt{n}}{\sqrt{n}}; \qquad \left(\frac{\sin n+3}{5}\right)^n; \qquad (3+\cos n)^n;$$

$$\frac{n}{n+1}\sin\frac{n\pi}{10}; \qquad \frac{1}{n^2}\log(1+2e^n); \qquad \frac{(kn)^n-n^{kn}}{(n-1)^{kn}+(n+k)!} \quad (k\in\mathbb{N}).$$

Esercizio 20. Calcolare, se esiste, il limite delle seguenti successioni.

$$\frac{2^n+n^2}{3^n+n^3}; \qquad \frac{(-2)^n+n^2}{3^n+n^3}; \qquad \frac{2^{\left[\frac{n}{2}\right]}+1}{2^n+n}; \qquad \frac{2^{\left[\frac{n}{2}\right]}+n^2}{2^{n/2}+n};$$

$$\frac{2^{n+2}+3^{n/2}}{2^n+3^{\frac{n+2}{2}}}; \qquad \alpha^n+\alpha^{2n} \quad (\alpha\in\mathbb{R}); \qquad \alpha^n+(-1)^n\alpha^{2n} \quad (\alpha\in\mathbb{R}); \qquad \frac{n^n+n^2}{n!+3^n};$$

$$\frac{(n+1)^n+n^4}{3^n n!+4^n}; \qquad (2-n)^n+(4n)!; \qquad \left(n!\sin\frac{1}{n^n}\right)^{1/n}; \qquad \sqrt[3]{n+1}-\sqrt[3]{n};$$

$$\sqrt[n]{\frac{(2n)!}{n!+1}}; \qquad \frac{\sqrt[n]{(2n)!}}{n}; \qquad \sqrt[n]{\frac{n^{n/2}}{2^n+1}}; \qquad \frac{\log n!}{n};$$

$$\frac{\log n!}{n^2}; \qquad \frac{\sum_{k=1}^n k^\alpha}{n} \quad (\alpha\in\mathbb{R}); \qquad \frac{\sum_{k=1}^n k^3}{n^3}; \qquad \frac{n^{n/2}}{n!};$$

$$\sqrt[n]{\binom{kn}{n}} \quad (k\in\mathbb{N}); \qquad \frac{n^{kn}}{(kn)!}; \qquad \frac{2^n}{\sqrt[n]{n!}}; \qquad \sqrt[n]{(n^2)!};$$

$$\sum_{k=n+1}^{2n}\frac{1}{k}.$$