Esercizio 4:

Secondo Preliminare, 13/12/2005

Calcolare
$$\int_{-\pi/2}^{\pi/2} \frac{\log(1+|\operatorname{sen} x|)}{(1+|\operatorname{sen} x|)^2} \cos x \, dx$$

Soluzione:

$$\frac{\log(1+|\operatorname{sen} x|)}{(1+|\operatorname{sen} x|)^2} \cdot \cos x$$ è una funzione pari in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Pertanto
$$\int_{-\pi/2}^{\pi/2} \frac{\log(1+|\operatorname{sen} x|)}{(1+|\operatorname{sen} x|)^2} \cos x \, dx = 2\int_0^{\pi/2} \frac{\log(1+|\operatorname{sen} x|)}{(1+|\operatorname{sen} x|)^2} \cos x \, dx$$

$$= 2\int_0^{\pi/2} \frac{\log(1+\operatorname{sen} x)}{(1+\operatorname{sen} x)^2} \cos x \, dx$$

$x \in (0, \frac{\pi}{2}) \Rightarrow \operatorname{sen} x > 0$

Facciamo la sostituzione $\operatorname{sen} x = t \Rightarrow d\operatorname{sen} x = dt \Rightarrow$

$\Rightarrow \cos x \, dx = dt$

Quindi
$$2\int_0^{\pi/2} \frac{\log(1+\operatorname{sen} x)}{(1+\operatorname{sen} x)^2} \cos x \, dx = 2\int_0^1 \frac{\log(1+t)}{(1+t)^2} dt =$$

$$= 2\int_0^1 \frac{1}{(1+t)^2} \log(1+t)\, dt = 2\int_0^1 f'(t)\, g(t)\, dt$$

dove $f'(t) = \frac{1}{(1+t)^2} \longrightarrow f(t) = -\frac{1}{1+t}$

$g(t) = \log(1+t) \longrightarrow g'(t) = \frac{1}{1+t}$.

Integro per parti

$$2\int_0^1 \frac{1}{(1+t)^2} \log(1+t)\, dt =$$

$$= 2\left\{ -\left[\frac{1}{1+t} \cdot \log(1+t)\right]_0^1 - \int_0^1 \frac{-1}{1+t} \cdot \frac{1}{1+t}\, dt \right\} =$$

$$= 2\left\{ -\left(\frac{\log 2}{2} - 0\right) + \int_0^1 \frac{1}{(1+t)^2}\, dt \right\} =$$

$$= -\log 2 + 2\int_0^1 \frac{1}{(1+t)^2}\, dt = -\log 2 + 2\left[\frac{-1}{1+t}\right]_0^1 =$$

$$= -\log 2 + 2\left(-\frac{1}{2} + 1\right) = -\log 2 + 1.$$