Esercizio 2 : Secondo Prelimin. 13/12/2005

Calcolare, se esiste, il seguente limite

$$\lim_{x\to 0^+} \frac{(1+\operatorname{tg} x)^x - (1+x)^x}{x^2(\cos\sqrt{x} - 1)}$$

Soluz:

$$\lim_{x\to 0^+} \frac{(1+\operatorname{tg} x)^x - (1+x)^x}{x^2(\cos\sqrt{x} - 1)} = \lim_{x\to 0^+} \frac{x}{\cos\sqrt{x}-1} \cdot \frac{(1+\operatorname{tg} x)^x - (1+x)^x}{x^3}$$

Tenendo conto che

$$\lim_{x\to 0^+} \frac{x}{\cos\sqrt{x}-1} = -2 \text{, si ha che}$$

$$\lim_{x\to 0^+} \frac{x}{\cos\sqrt{x}-1} \cdot \frac{(1+\operatorname{tg} x)^x - (1+x)^x}{x^3} =$$

$$= -2 \lim_{x\to 0^+} \frac{(1+\operatorname{tg} x)^x - (1+x)^x}{x^3} =$$

$$= -2 \lim_{x\to 0^+} \frac{e^{x\log(1+\operatorname{tg} x)} - e^{x\log(1+x)}}{x^3}.$$

Ora: se $y \to 0$

$$\log(1+y) = y - \frac{y^2}{2} + \frac{y^3}{3} + o(y^3)$$

$$\operatorname{tg} y = y + \frac{y^3}{3} + o(y^4) \quad ; \quad e^y = 1 + y + \frac{y^2}{2} + o(y^2)$$

PRIMA proposta di soluzione (sconsigliata)

$$\log(1+\operatorname{tg} x) = \operatorname{tg} x - \frac{(\operatorname{tg} x)^2}{2} + \frac{(\operatorname{tg} x)^3}{3} + o((\operatorname{tg} x)^3)$$

$$= x + \frac{x^3}{3} + o(x^4) - \frac{1}{2}\left(x + \frac{x^3}{3} + o(x^4)\right)^2 + \frac{1}{3}\cdot\left(x + \frac{x^3}{3} + o(x^4)\right)^3 + o(x^3)$$

$$= x + \frac{x^3}{3} + o(x^4) - \frac{x^2}{2} + o(x^3) + \frac{1}{3}\cdot x^3 + o(x^3) =$$

$$= x - \frac{x^2}{2} + \frac{2}{3}x^3 + o(x^3)$$

e

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)$$

Pertanto:

$$e^{x\log(1+\operatorname{tg} x)} - e^{x\log(1+x)} =$$

$$= e^{x\left[x - \frac{x^2}{2} + \frac{2}{3}x^3 + o(x^3)\right]} - e^{x\left[x - \frac{x^2}{2} + \frac{x^3}{3} + o(x^3)\right]} =$$

$$= e^{x^2 - \frac{x^3}{2} + \frac{2}{3}x^4 + o(x^4)} - e^{x^2 - \frac{x^3}{2} + \frac{x^4}{3} + o(x^4)} = \quad (x\to 0)$$

$$= 1 + \left(x^2 - \frac{x^3}{2} + \frac{2}{3}x^4 + o(x^4)\right) + \frac{1}{2}\left(x^2 - \frac{x^3}{2} + \frac{2}{3}x^4 + o(x^4)\right)^2 + o(x^4)$$

$$- \left\{1 + \left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} + o(x^4)\right) + \frac{1}{2}\left(x^2 - \frac{x^3}{2} + \frac{x^4}{3} + o(x^4)\right)^2 + o(x^4)\right\}$$

$$= \cancel{1} + \cancel{x^2} - \cancel{\frac{x^3}{2}} + \frac{2}{3}x^4 + o(x^4) + \frac{x^4}{2} + o(x^4) -$$

$$- \left\{\cancel{1} + \cancel{x^2} - \cancel{\frac{x^3}{2}} + \frac{x^4}{3} + o(x^4) + \frac{x^4}{2} + o(x^4)\right\} =$$

$$= \left\{\frac{2}{3} + \frac{1}{2} - \frac{1}{3} + \frac{1}{2}\right\} x^4 + o(x^4) = \left(\frac{1}{3} + 1\right)x^4 + o(x^4)$$

$$= \frac{4}{3}x^4 + o(x^4).$$

Pertanto:

$$\lim_{x\to 0^+} \frac{(1+\operatorname{tg} x)^x - (1+x)^x}{x^2(\cos\sqrt{x} - 1)} = -2 \lim_{x\to 0} \frac{\frac{4}{3}x^4 + o(x^4)}{x^3}$$

$$= 0$$

SECONDA proposta di soluzione:

$$\log(1+\operatorname{tg} x) = \log(1 + x + o(x^2)) = x + o(x^2) - \frac{1}{2}(x + o(x^2))^2 =$$

$$= x - \frac{x^2}{2} + o(x^2)$$