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Sol: (3b)

Verifica di $\lim_{x \to 1} \frac{x}{(x-1)^2} = +\infty$

Sia $M > 0$. Allora se $x \neq 1$ si ha

$$\frac{x}{(x-1)^2} > M \iff x > M(x-1)^2 \iff$$

$$\iff Mx^2 - 2Mx - x + M < 0 \iff Mx^2 - x(1+2M) + M < 0$$

$$\Delta = (1+2M)^2 - 4M^2 = 1 + 4M > 0$$

Allora $Mx^2 - x(1+2M) + M < 0 \iff$

$$\iff x \in \left(\frac{1+2M-\sqrt{1+4M}}{2M}, \frac{1+2M+\sqrt{1+4M}}{2M} \right)$$

$$\iff x \in \left(1 - \frac{-1+\sqrt{1+4M}}{2M}, 1 + \frac{1+\sqrt{1+4M}}{2M} \right)$$

Quindi se $0 < |x-1| < \min\left\{ \frac{-1+\sqrt{1+4M}}{2M}, \frac{1+\sqrt{1+4M}}{2M} \right\}$,

cioè se $0 < |x-1| < \frac{-1+\sqrt{1+4M}}{2M}$, si ha che

$$\frac{x}{(x-1)^2} > M.$$

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Es4: Calcolare, se esiste, il seguente limite:

$$\lim_{n \to +\infty} n \left(\frac{e^{n!} + 5n^{2n}}{e^{n!} - 3n^{2n}} - e^{1/n} \right)$$

Sol: $\lim_{n \to +\infty} \frac{e^{n!}}{n^{2n}} = +\infty$.

ciò segue dal criterio del rapporto, infatti, posto

$$a_n := \frac{e^{n!}}{n^{2n}}, \text{ si ha: } \frac{a_{n+1}}{a_n} = \frac{e^{(n+1)!}}{(n+1)^{2n+2}} \cdot \frac{n^{2n}}{e^{n!}} =$$

$$= \frac{e^{(n+1)\cdot n!}}{e^{n!}} \cdot \frac{n^{2n}}{(n+1)^{2n} \cdot (n+1)^2} = \left(e^n\right)^{n!} \cdot \left(\frac{n}{n+1}\right)^{2n} \cdot \frac{1}{(n+1)^2} \geq$$

$$\geq \underbrace{\frac{e^n}{(n+1)^2}}_{\to +\infty} \cdot \underbrace{\frac{1}{\left(1+\frac{1}{n}\right)^{2n}}}_{\to \frac{1}{e^2}} \xrightarrow[n \to +\infty]{} +\infty$$

Ora:

$$n \left(\frac{e^{n!} + 5n^{2n}}{e^{n!} - 3n^{2n}} - e^{1/n} \right) = n \left(\frac{1 + 5\frac{n^{2n}}{e^{n!}}}{1 - 3\frac{n^{2n}}{e^{n!}}} - e^{1/n} \right) =$$

$$= n \left(\frac{1 + 5\frac{n^{2n}}{e^{n!}}}{1 - 3\frac{n^{2n}}{e^{n!}}} - 1 + 1 - e^{1/n} \right) =$$

$$= n \left(8 \, \frac{\frac{n^{2n}}{e^{n!}}}{1 - 3\frac{n^{2n}}{e^{n!}}} \right) + n \left(1 - e^{1/n} \right) =$$